Introduction to SIR Modeling

13th Annual Mathematical Modeling and Public Health Workshop

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Slides adapted from Stephen Kissler

HARVARD T.H. CHAN SCHOOL OF PUBLIC HEALTH

CENTER for COMMUNICABLE DISEASE DYNAMICS

Infectious disease burden

Economic burden

US$ 8 trillion

Health burden

168 million

disability-adjusted life years lost

* of just 8 infectious diseases in a single year (HIV/AIDS, malaria, measles, hepatitis, dengue fever, rabies, tuberculosis and yellow fever)

Institute of Labour Economics, 2020

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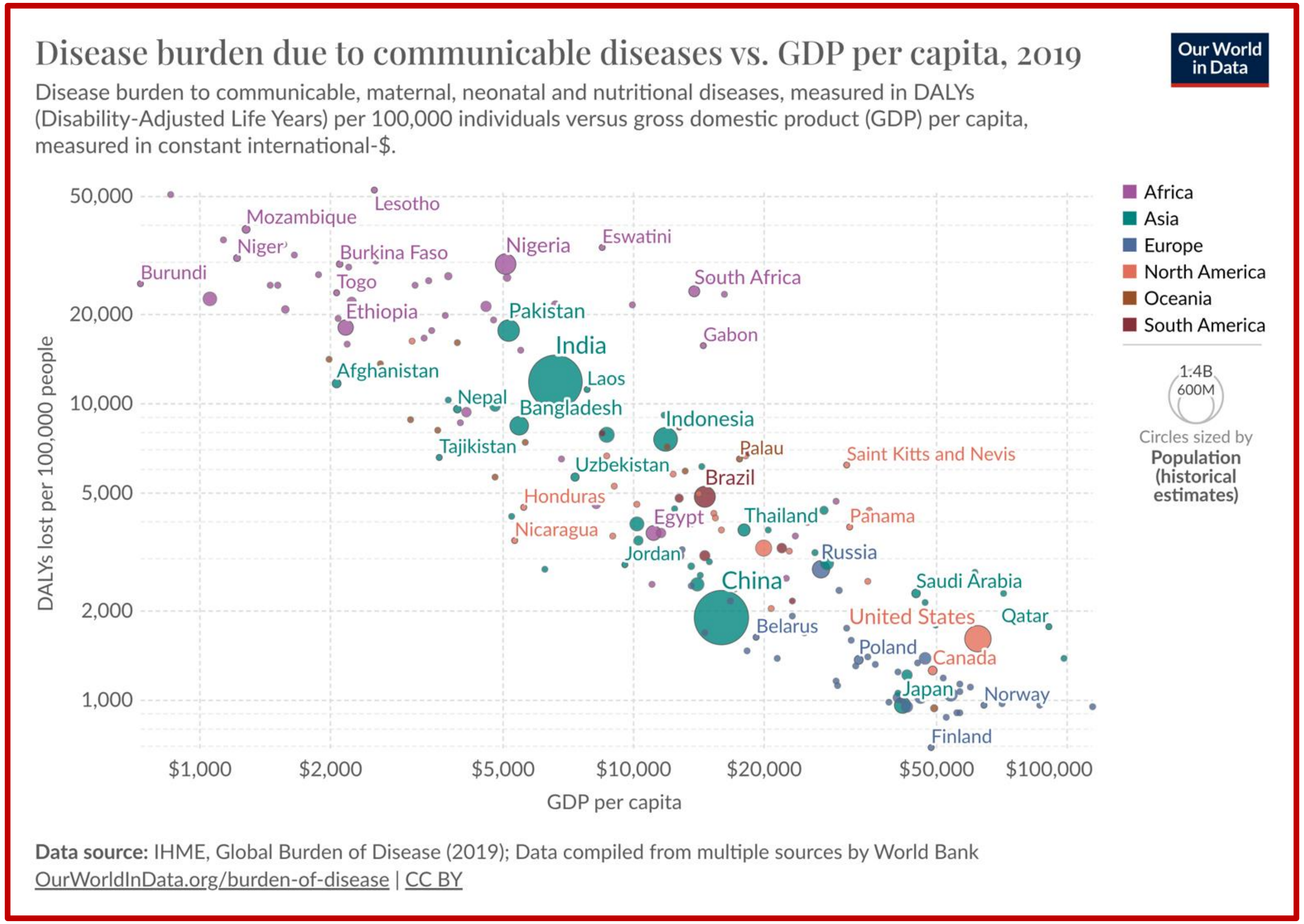
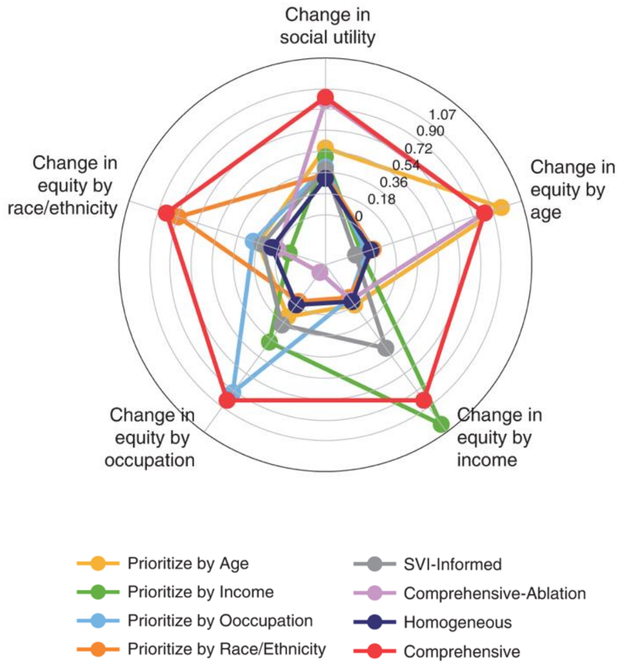
Institute of Labour Economics, 2020



Mathematical models to achieve public health goals

Prevention

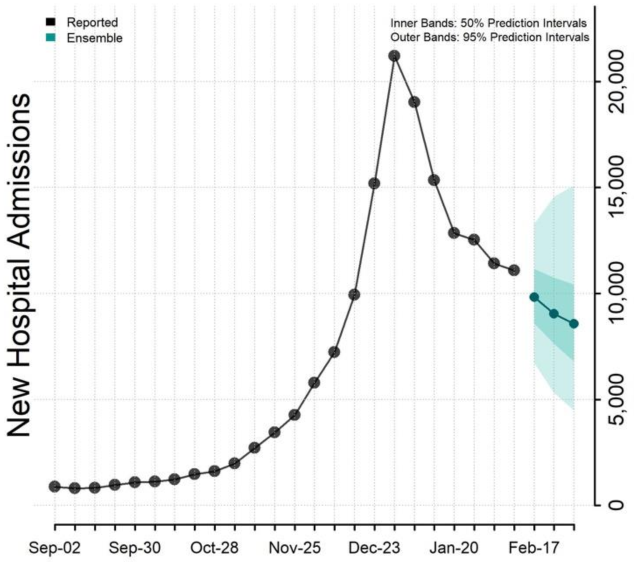
Which vaccine allocation strategy is most equitable?



Chen *et al.* (2022)

Prediction

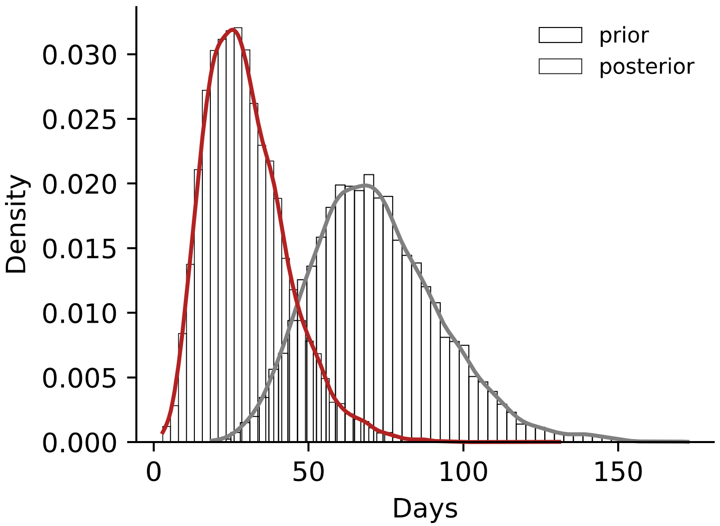
How many hospitalizations?



CDC, FluSight: Flu Forecasting

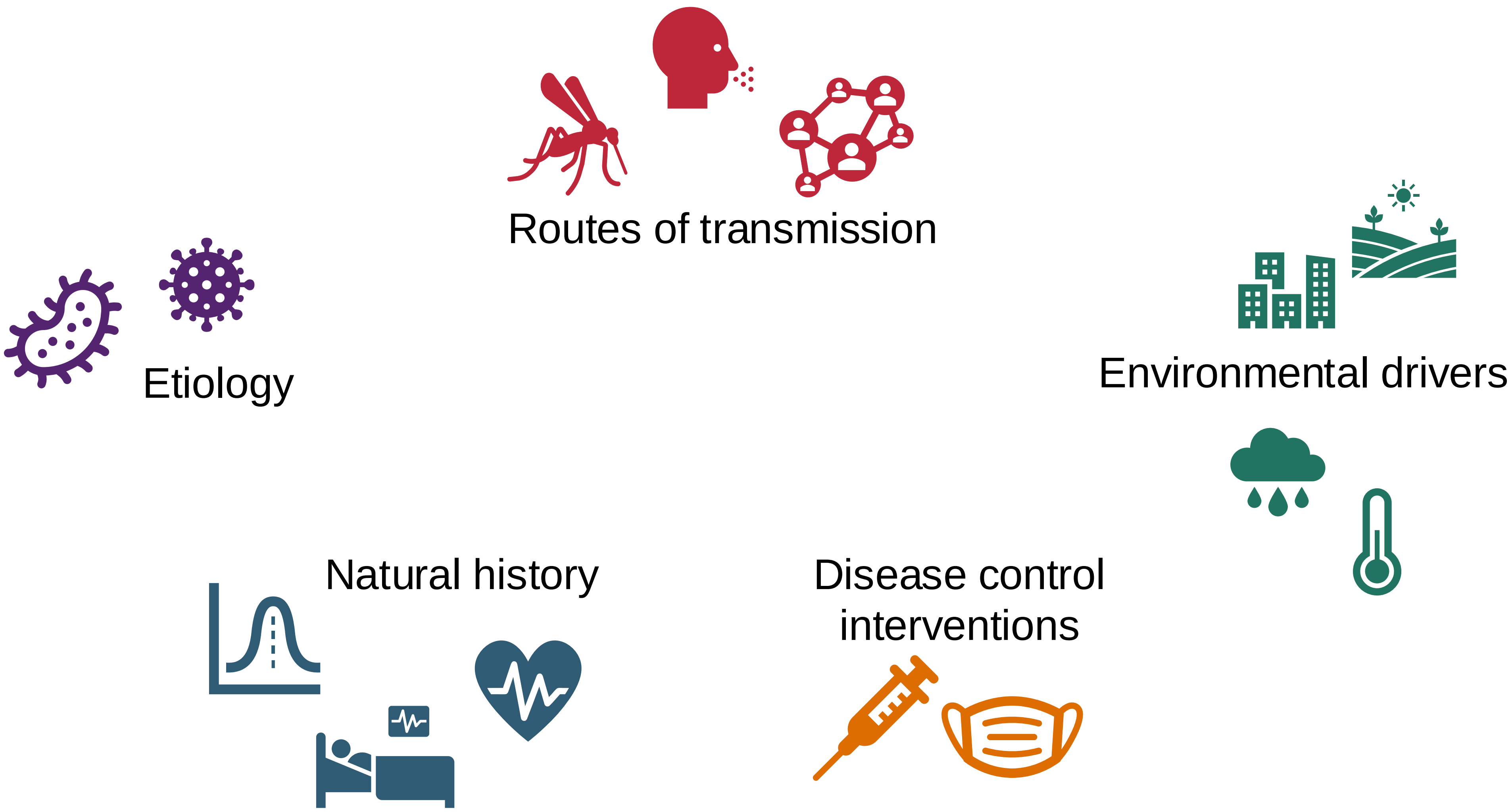
Understanding

How long is the latent* period?



*Latent period = time between infection and onset of symptoms

Characterizing infectious diseases



Malaria

Anopheles

Image: Wikipedia

- Active between sunset and sunrise
- Breed in natural bodies of water
- Multiple hosts

Insecticide treated bed nets

Image: USAID

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Anopheles

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Insecticide treated bed nets

Image: USAID

Dengue Fever

Aedes

Image: Wikipedia

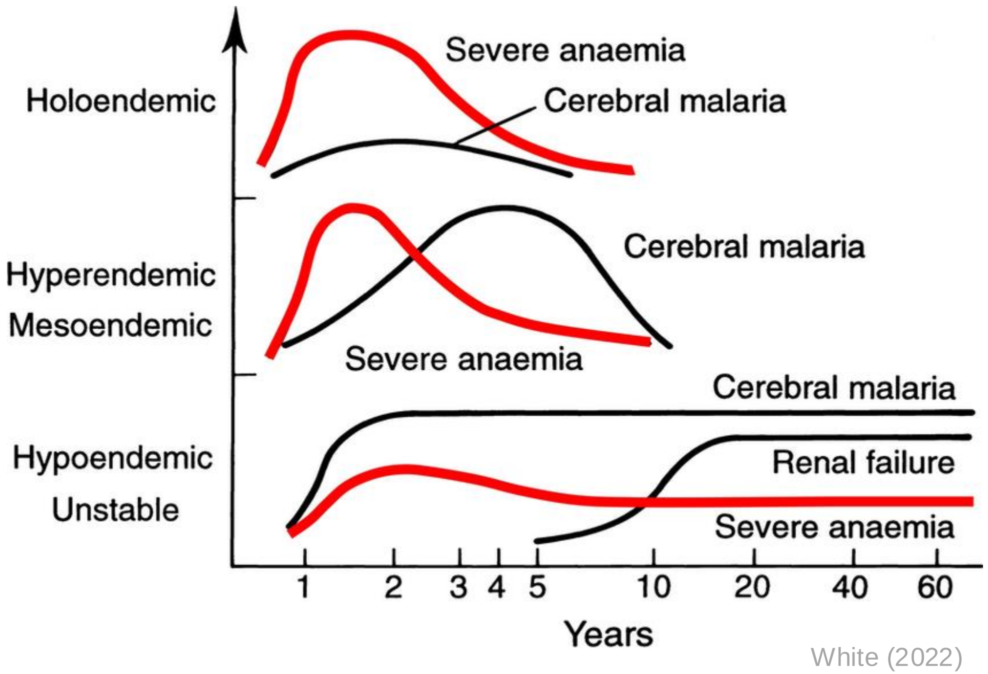
- Daytime feeders
- Highly domesticated
- Human is preferred host

Removal of open water containers

Images: Municipal Government of Acerburgo, BR; FAPTO

Malaria

Prior infection conveys some protection against reinfection and severe outcomes



White (2022)

Malaria

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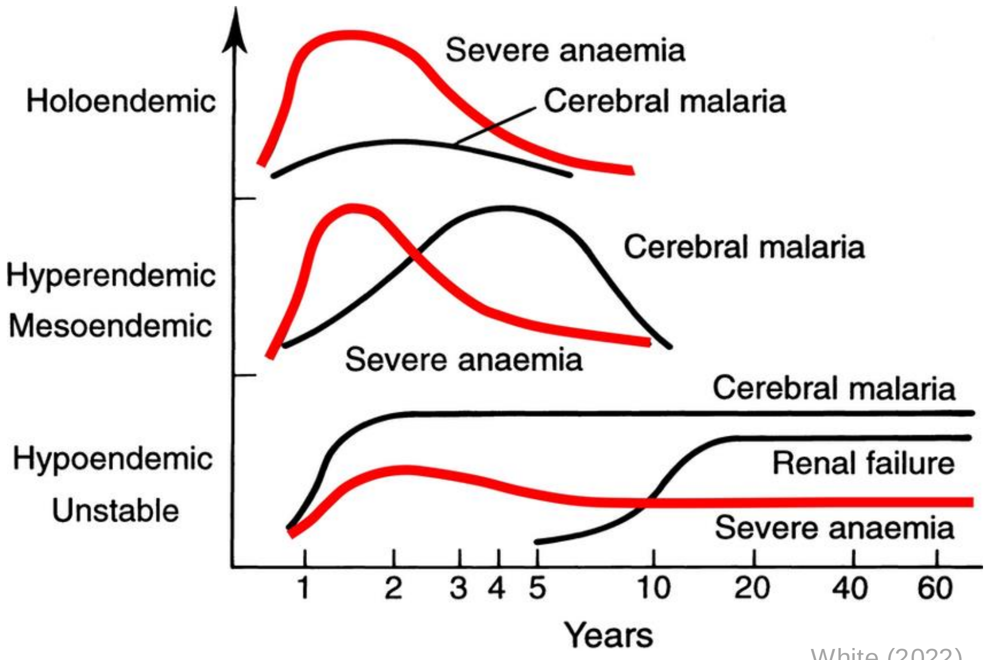
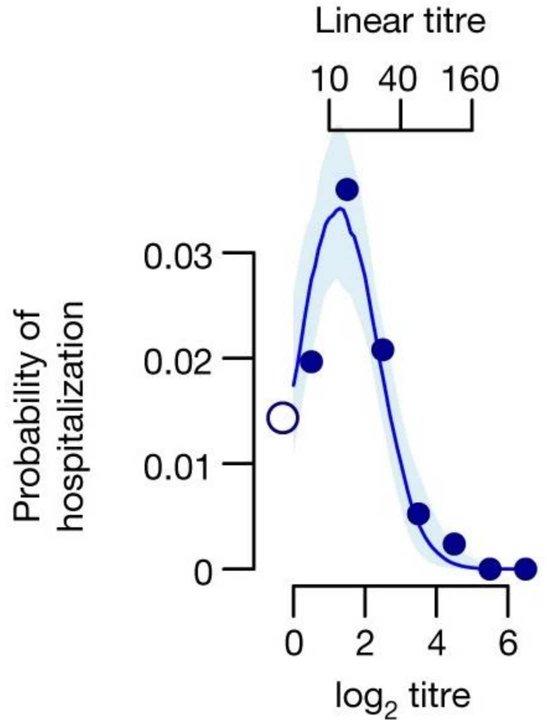


White (2022)

Dengue Fever

Prior infection increases the risk of severe dengue

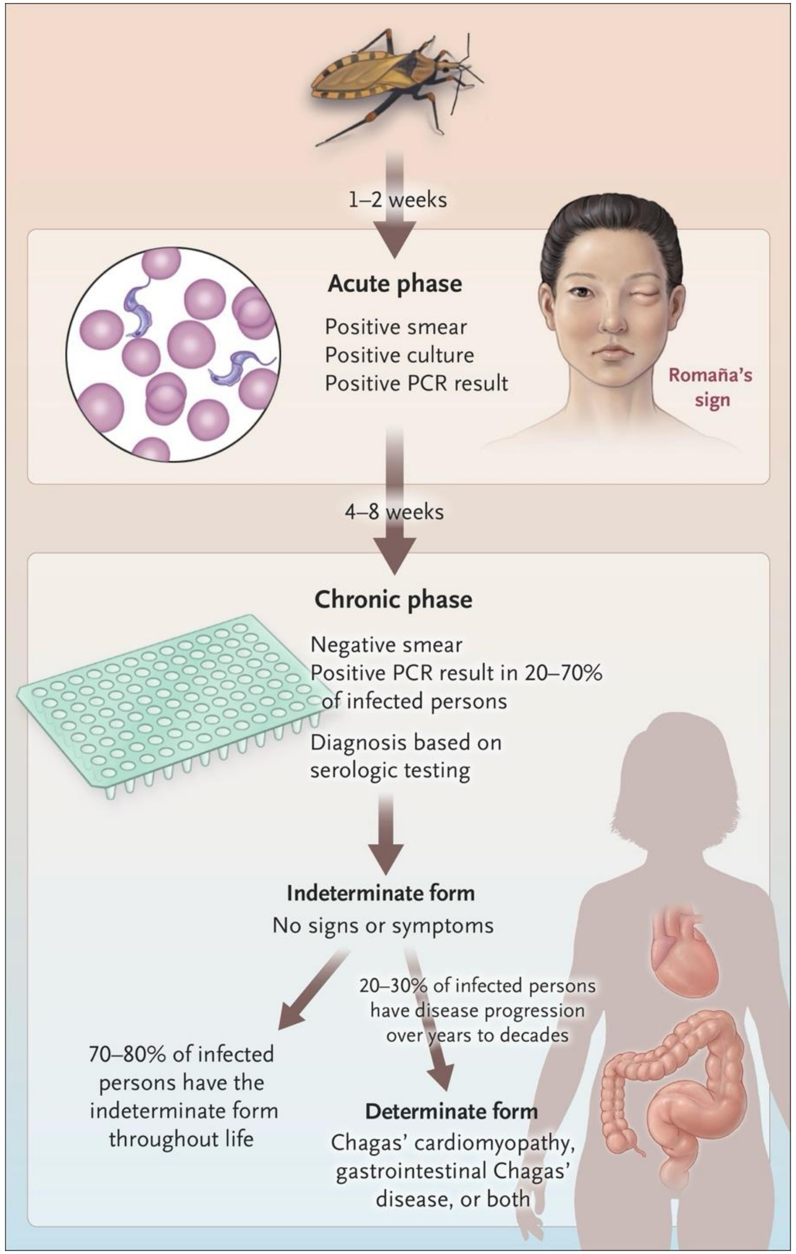


Salje *et al.* (2018)

Chagas Disease

Carlos Chagas
(1879 – 1934)

Image: Casa de Oswaldo Cruz

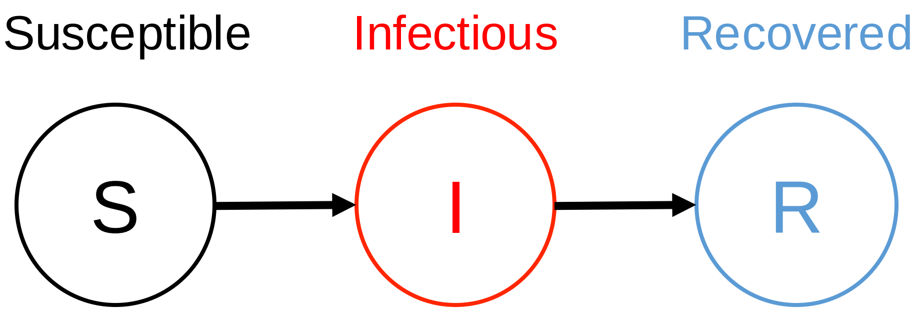


Vector control

Image: PAHO

Bern (2015)

Translating a disease process into a mathematical model


Susceptible
Infectious
Recovered
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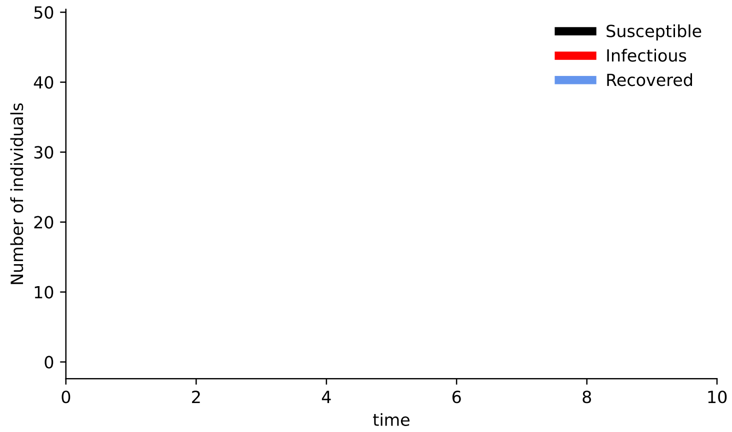
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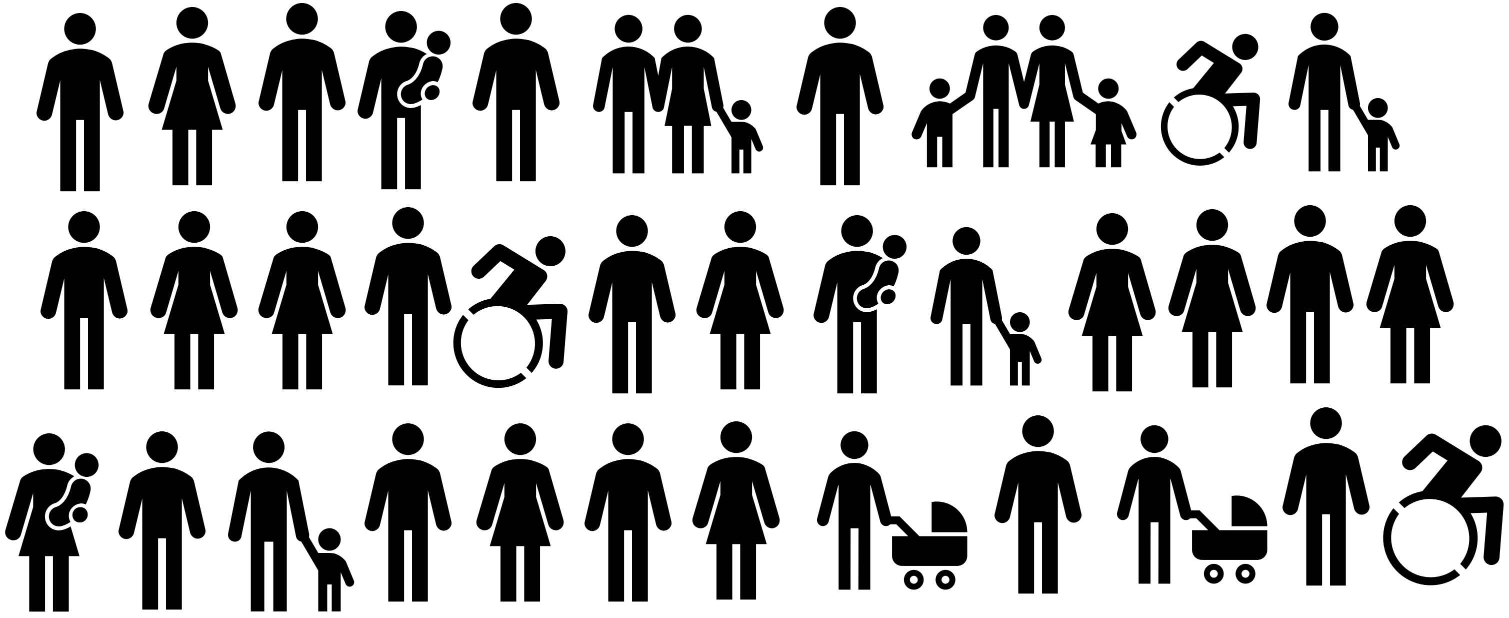

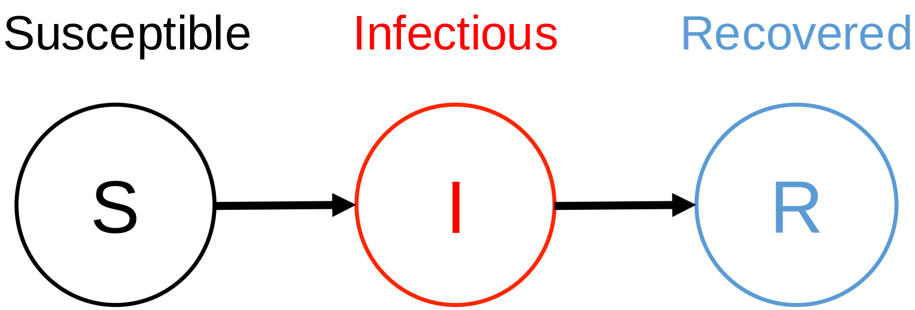
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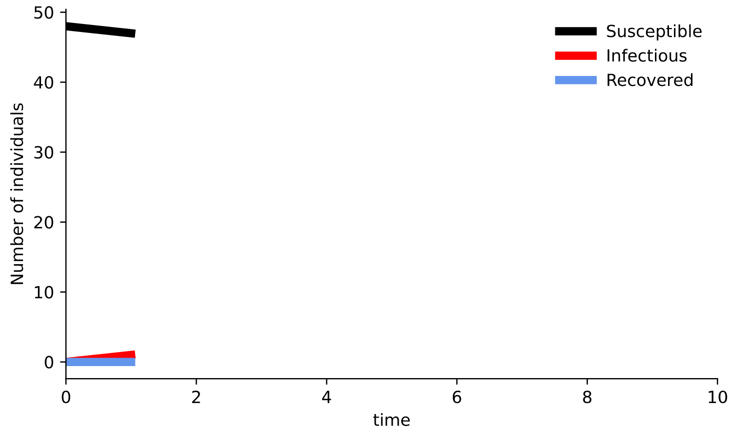
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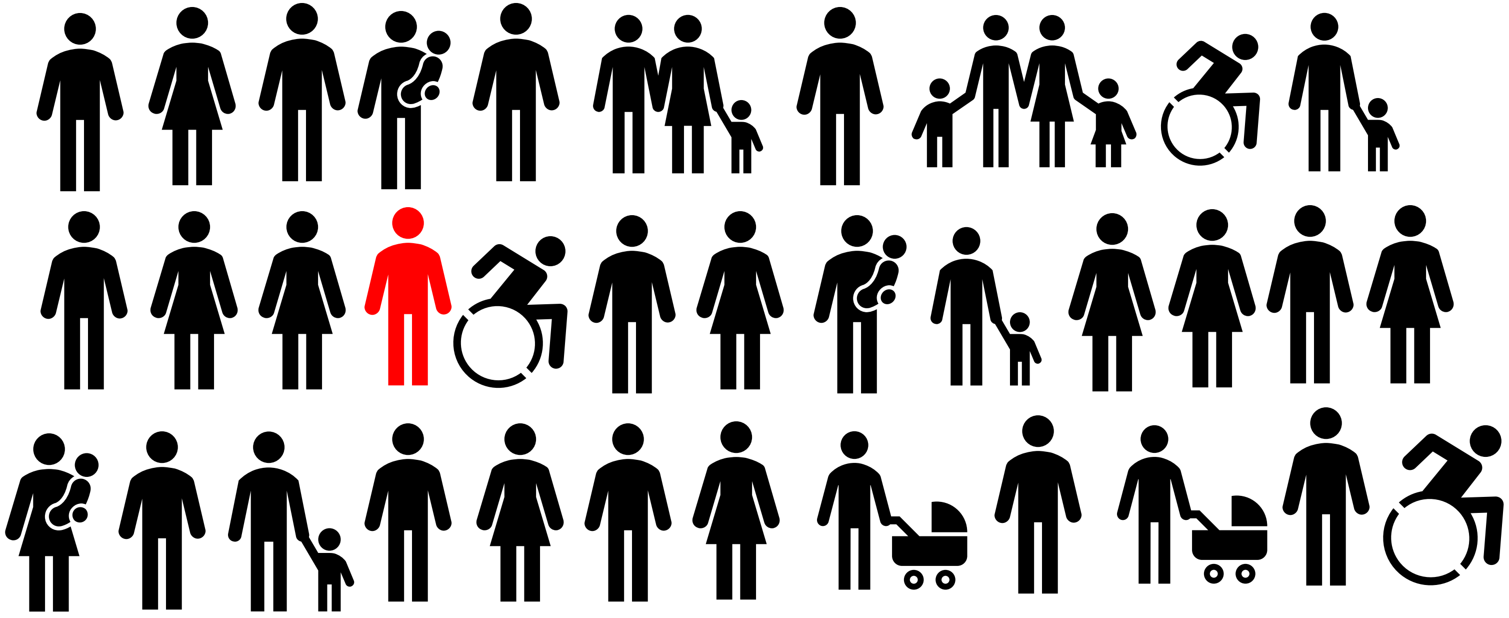

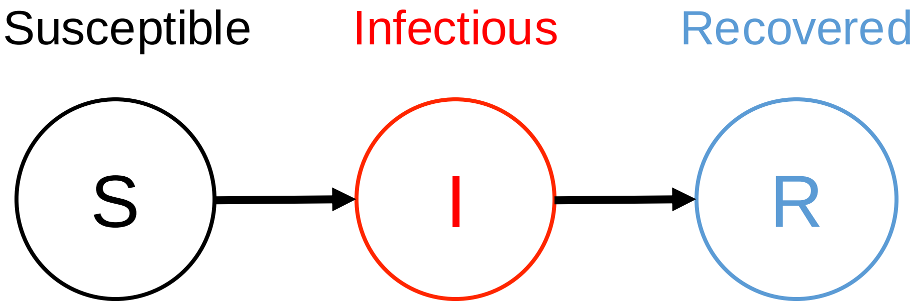
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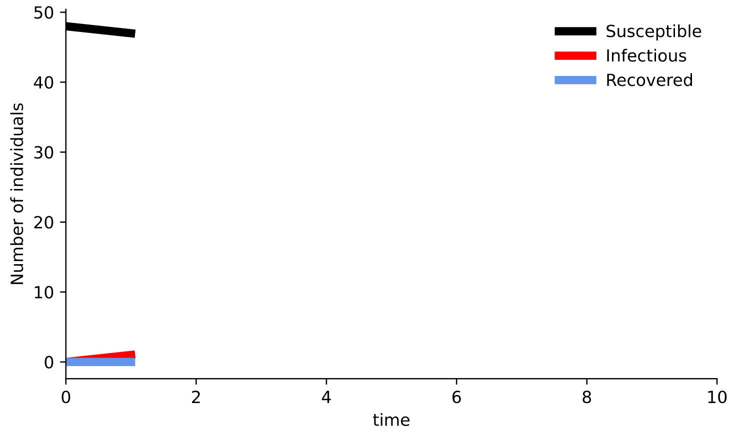
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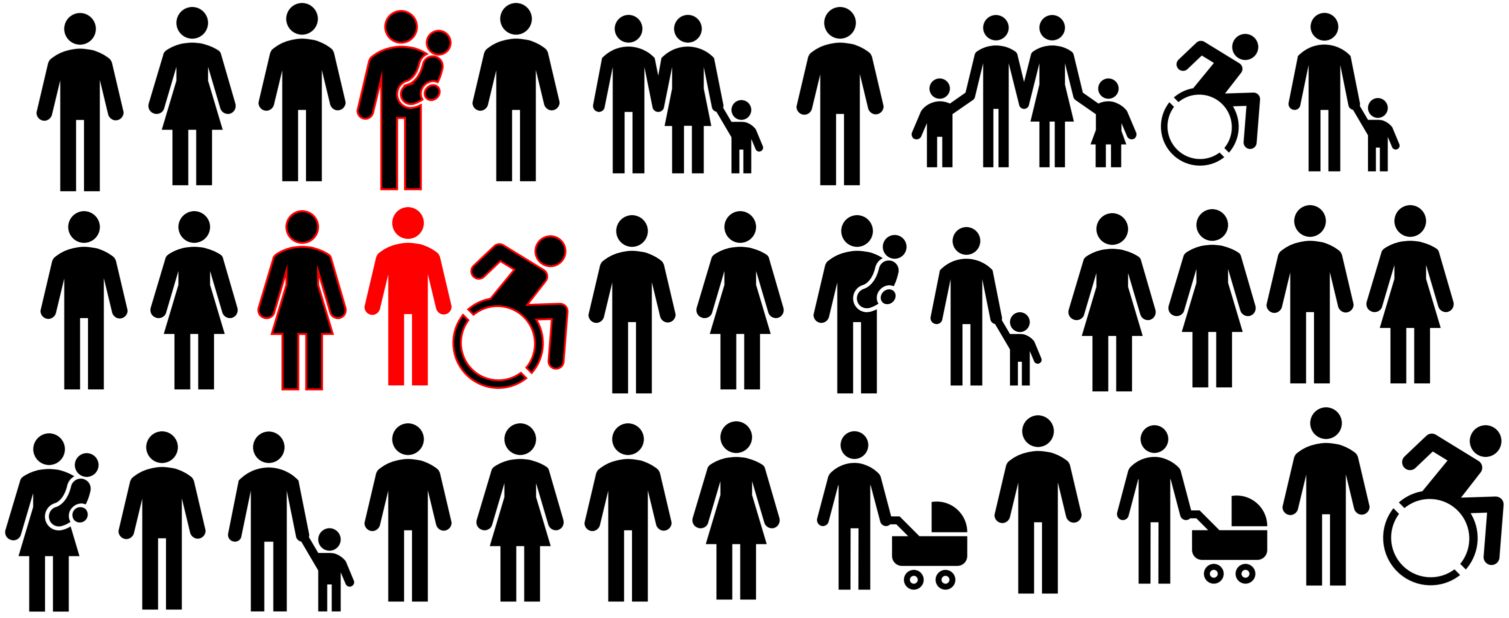

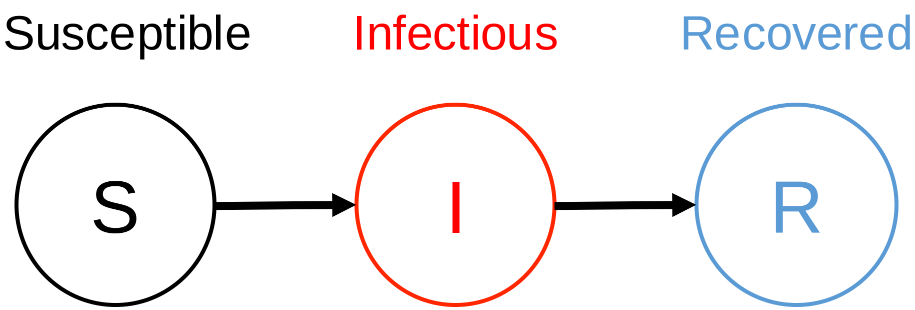
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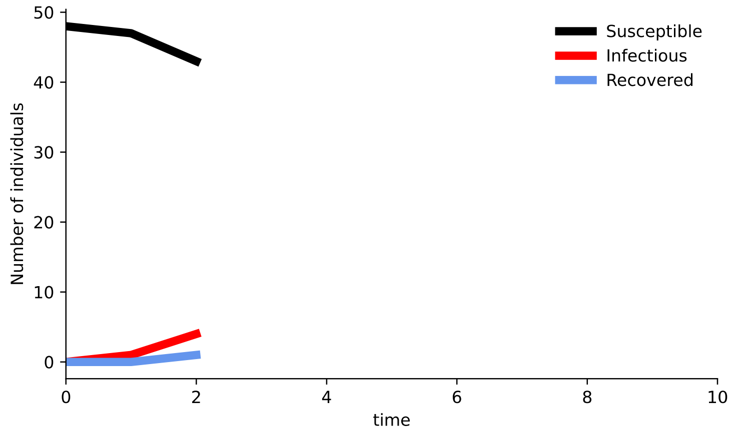
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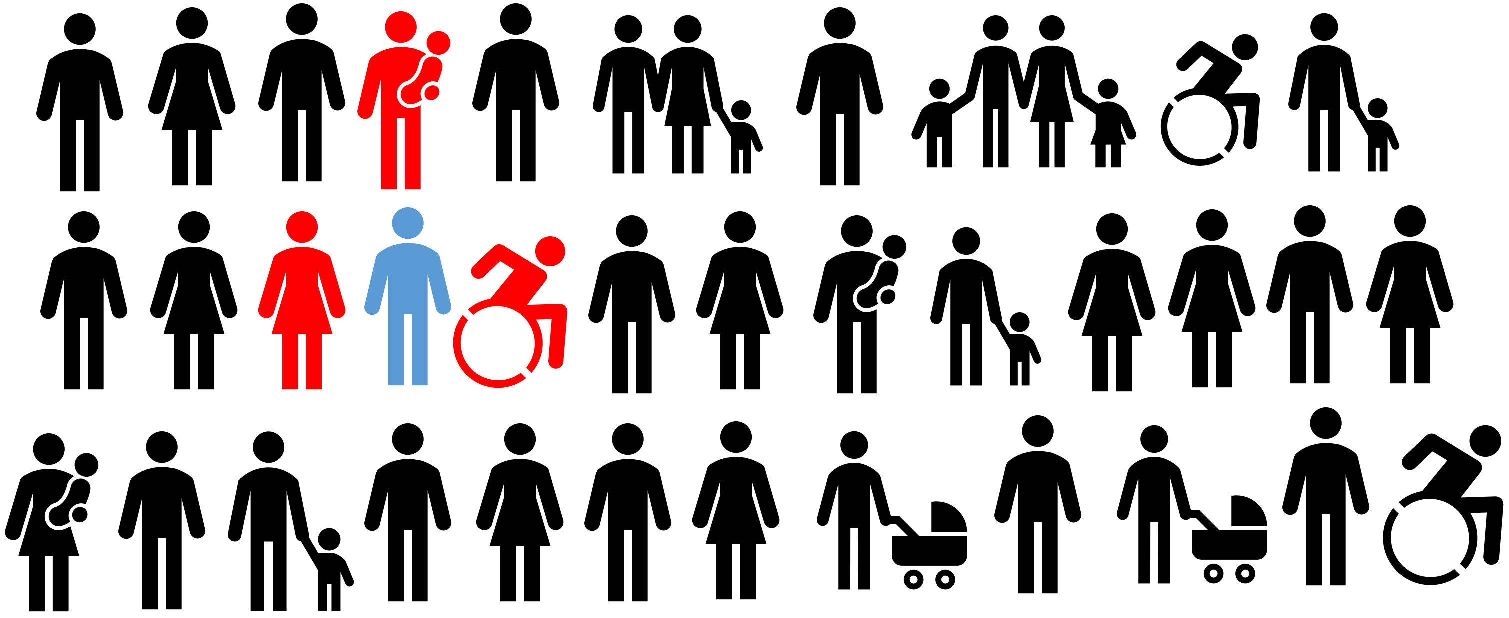

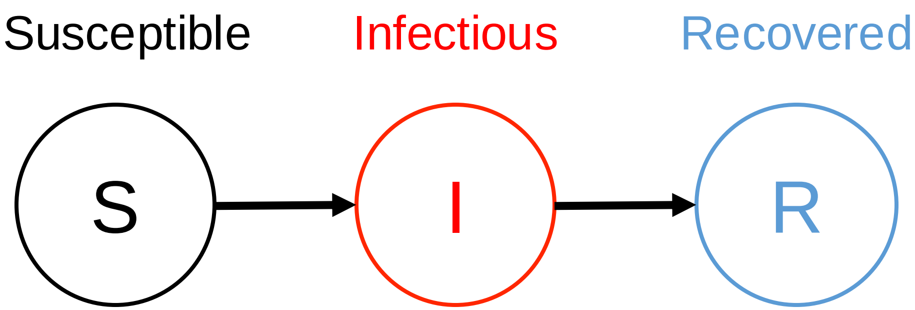
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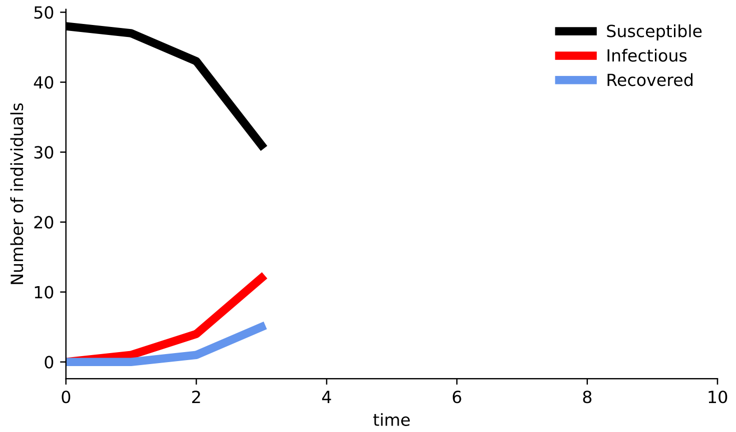
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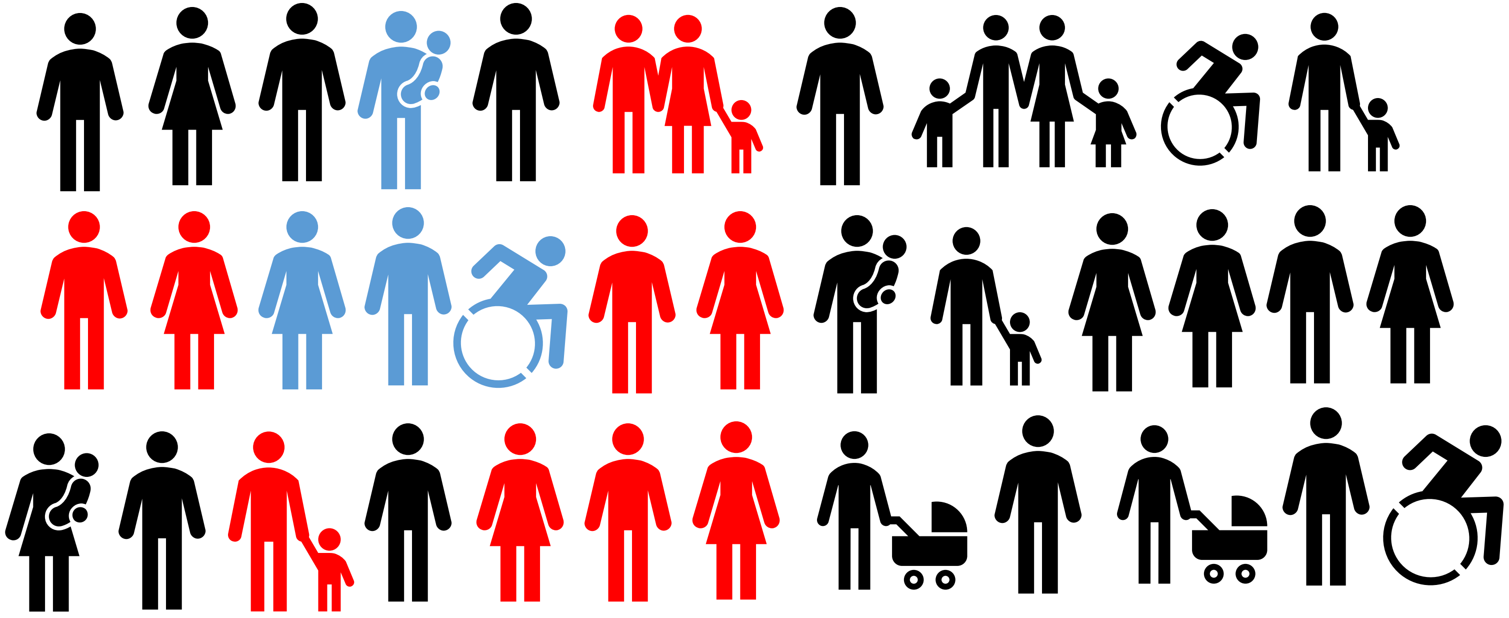

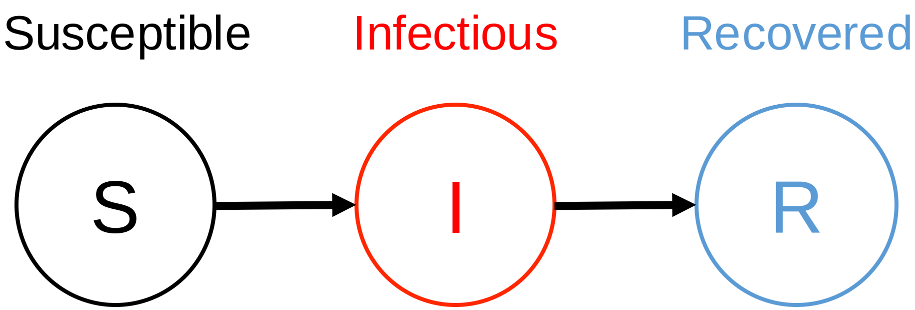
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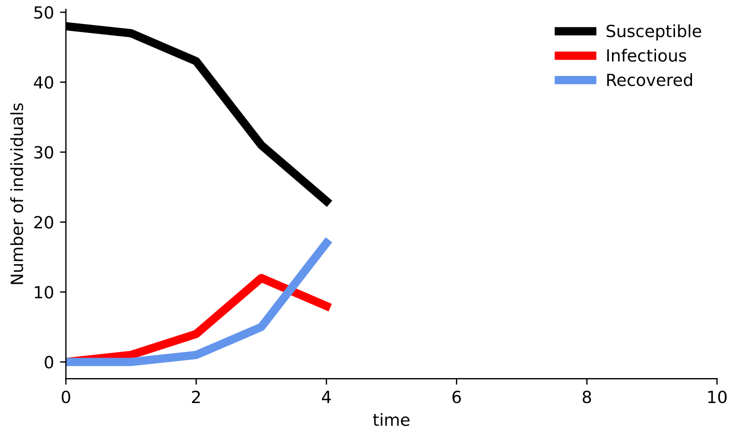
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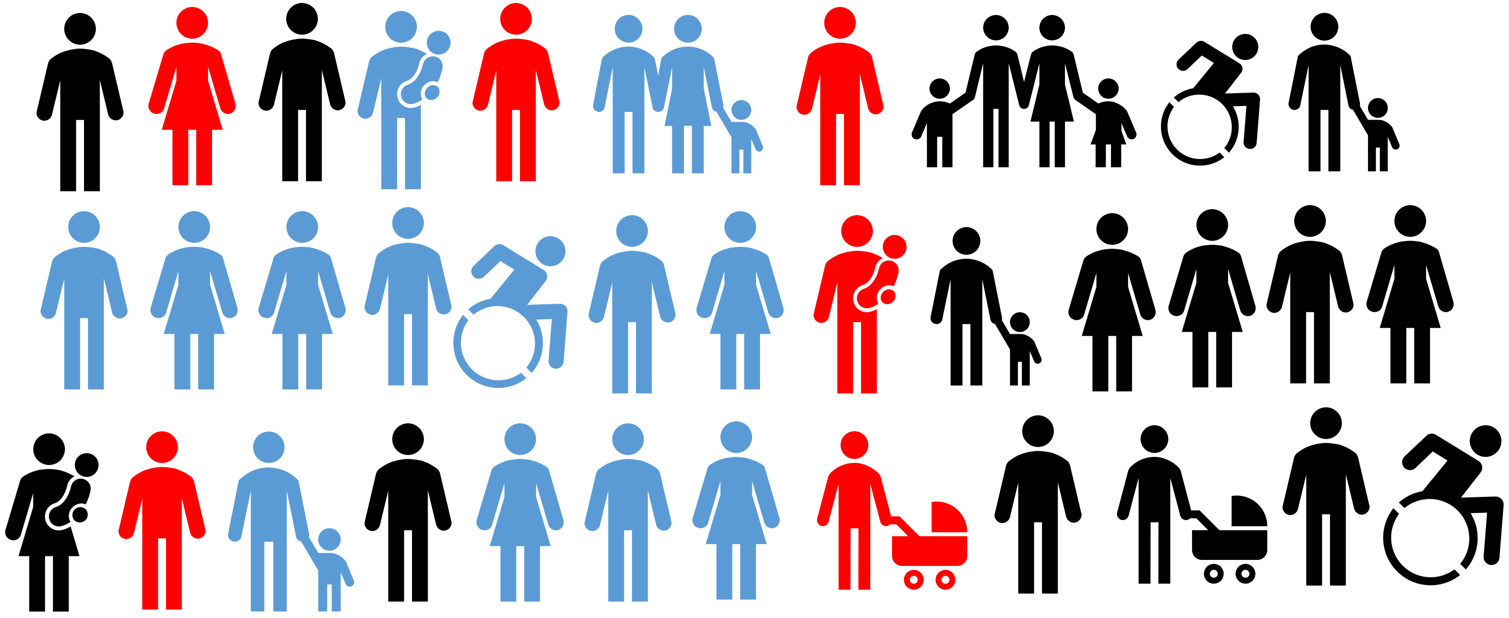

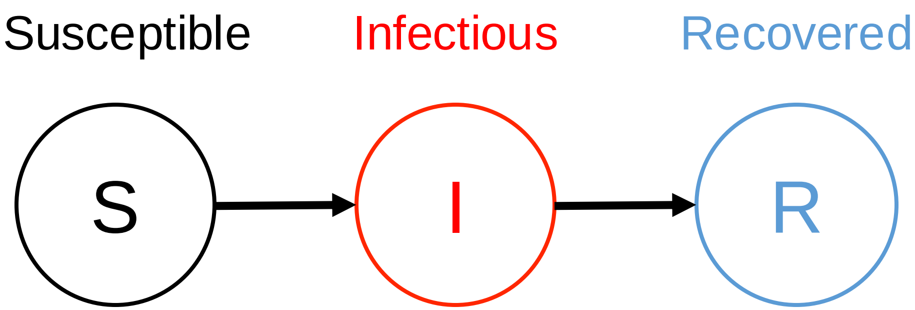
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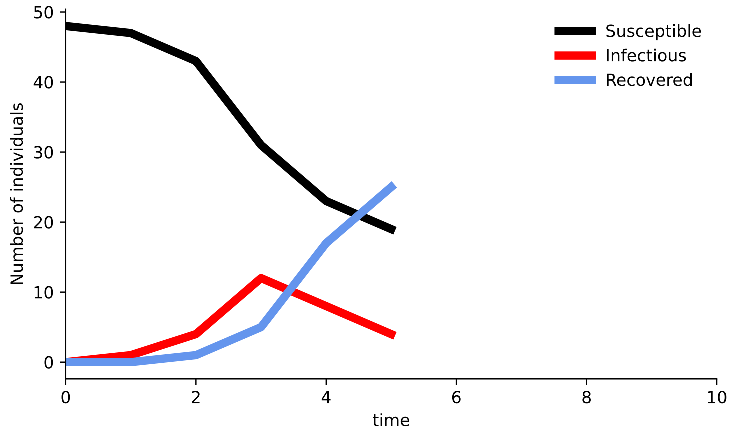
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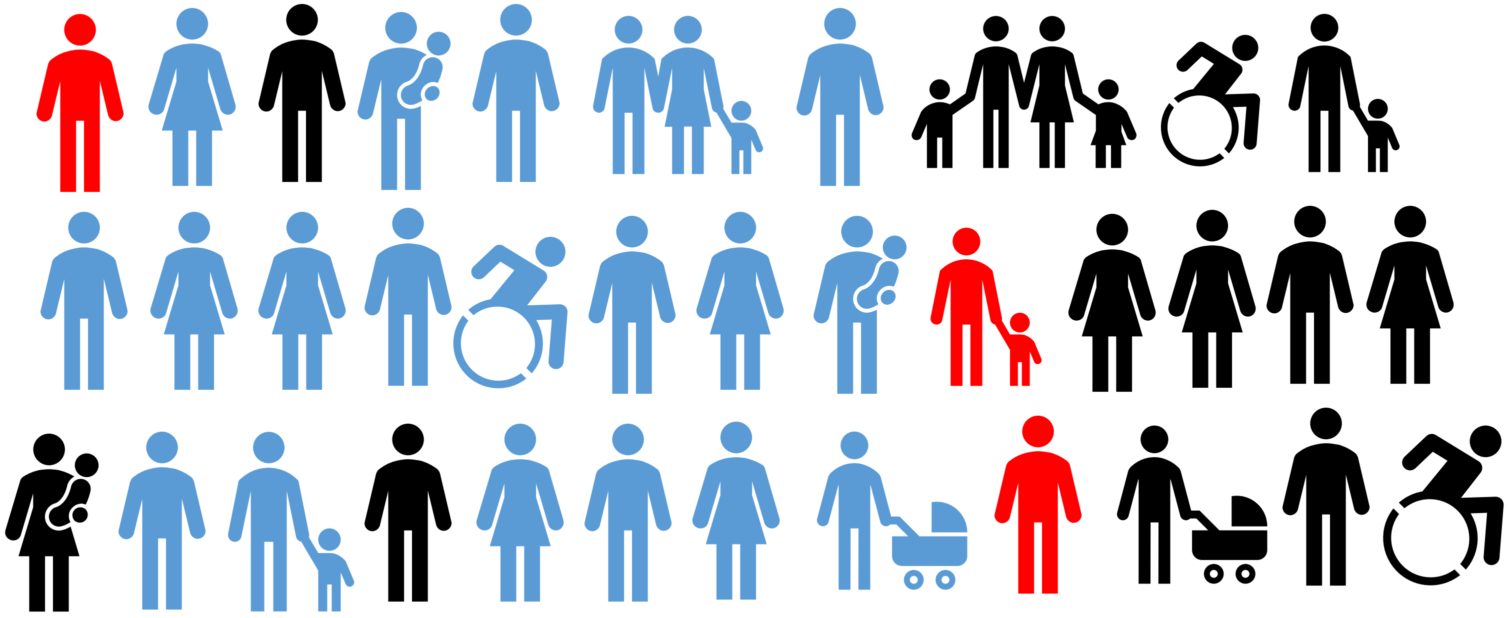

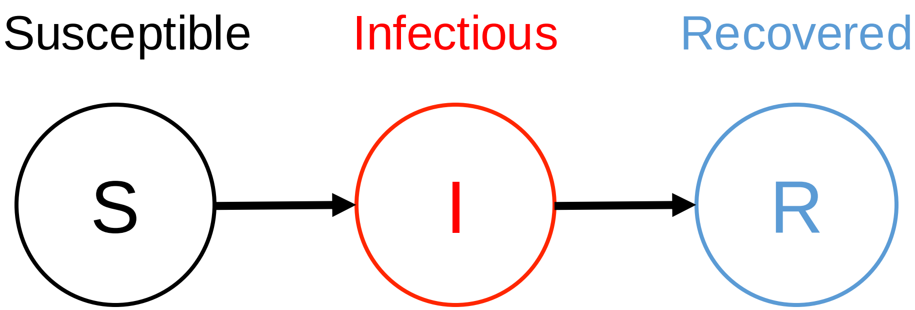
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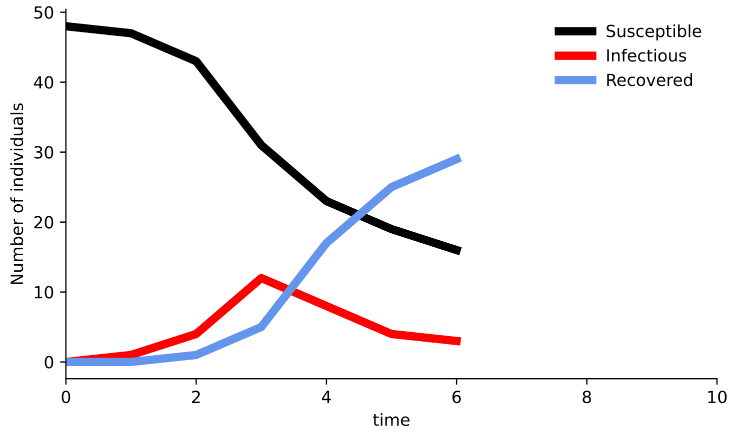
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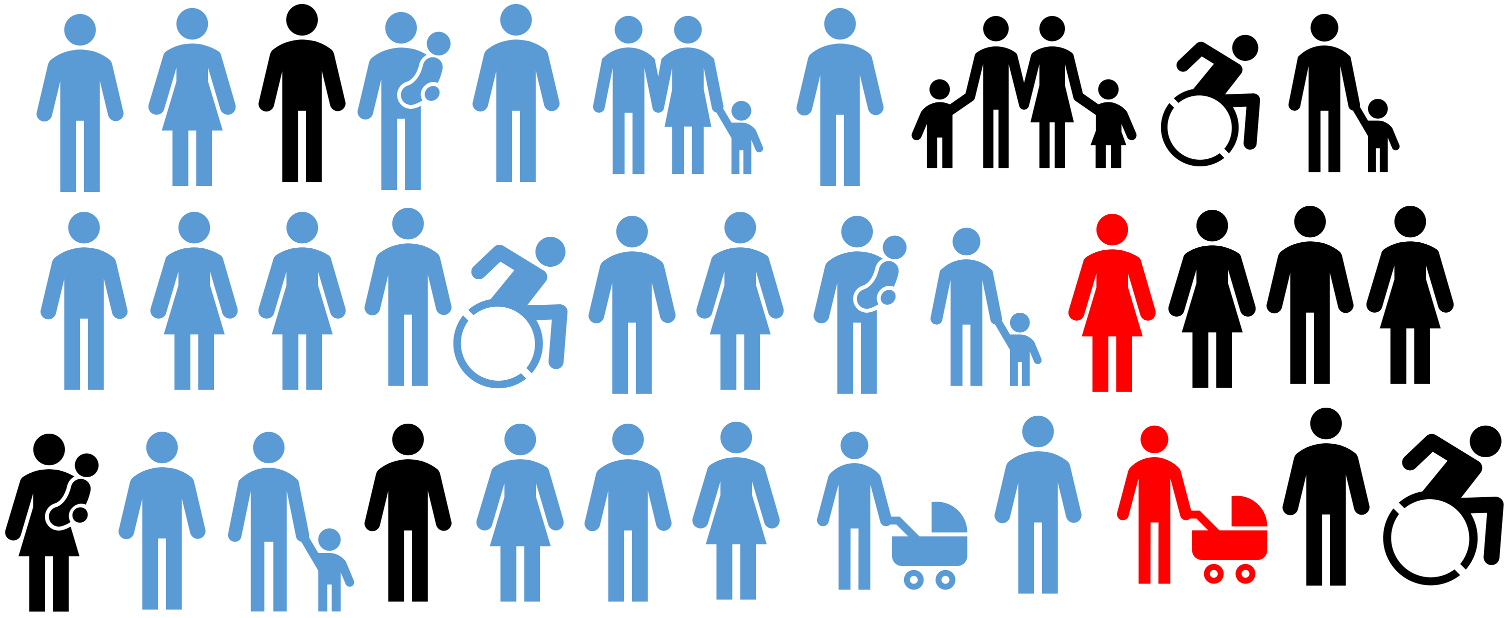

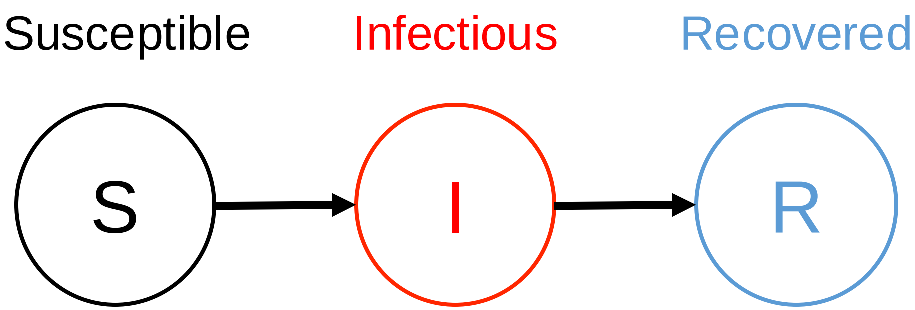
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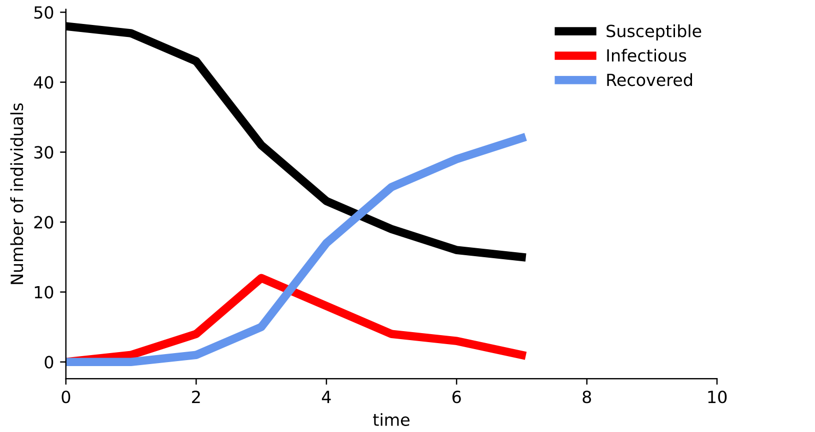
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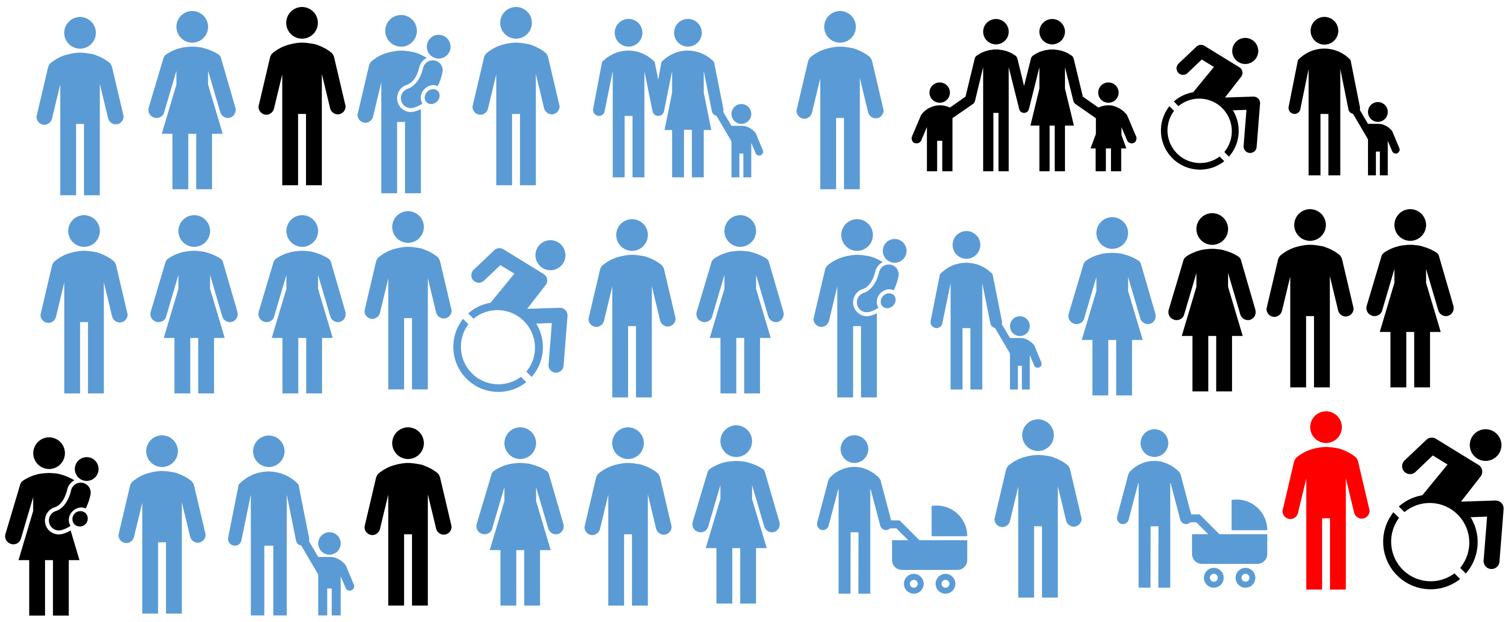

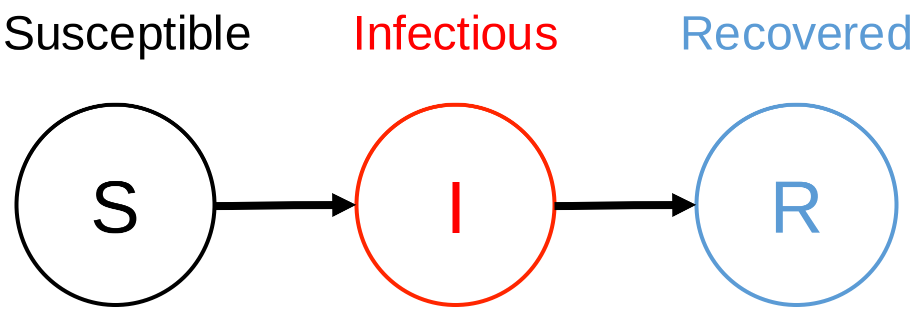
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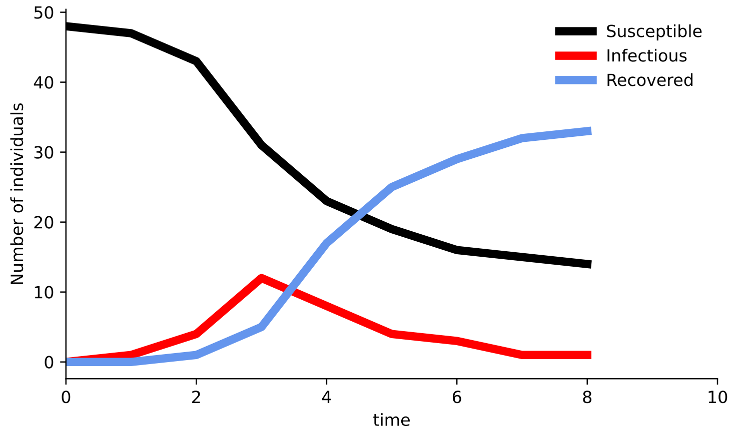
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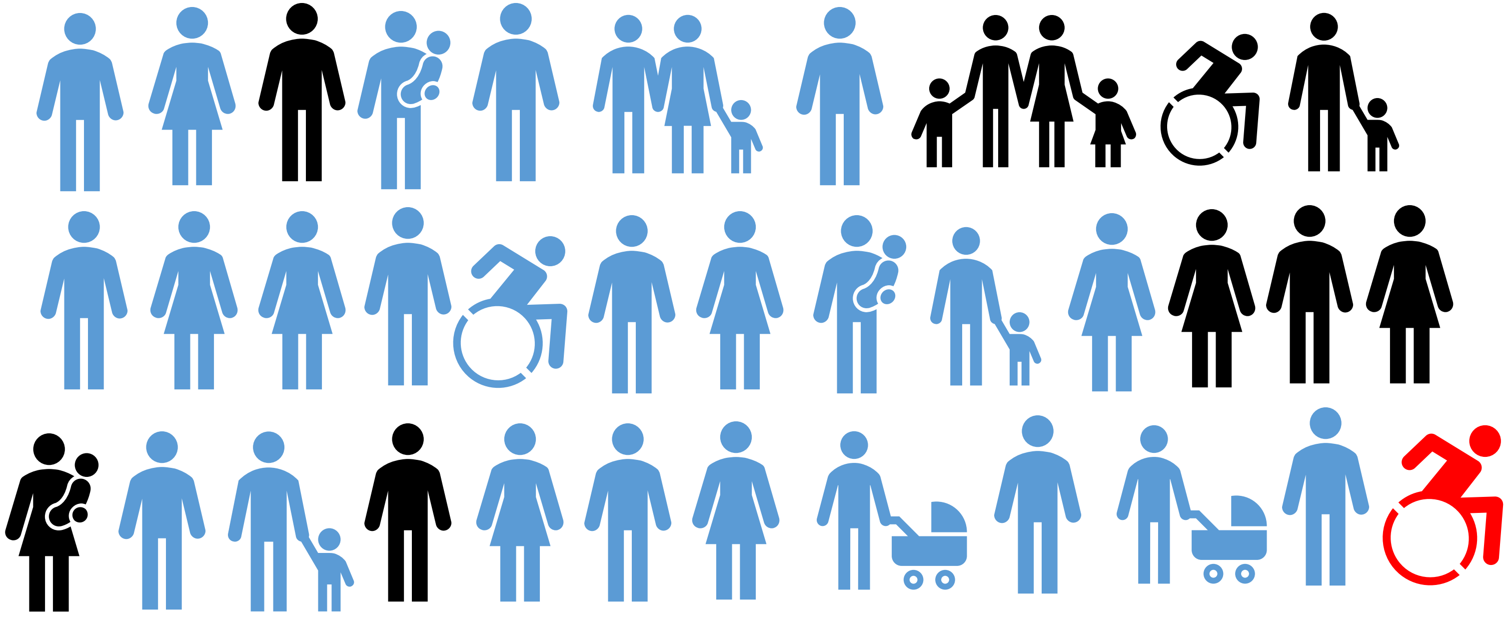

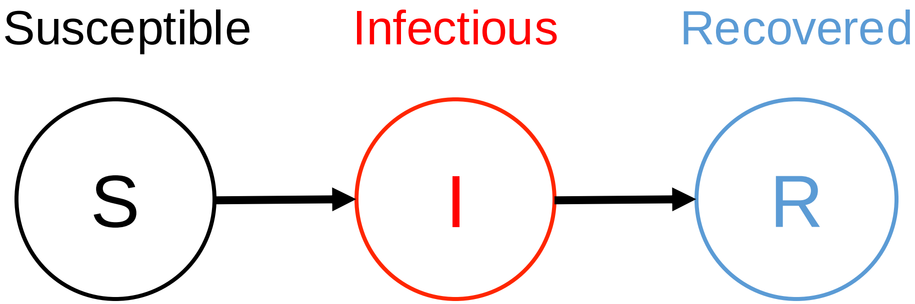
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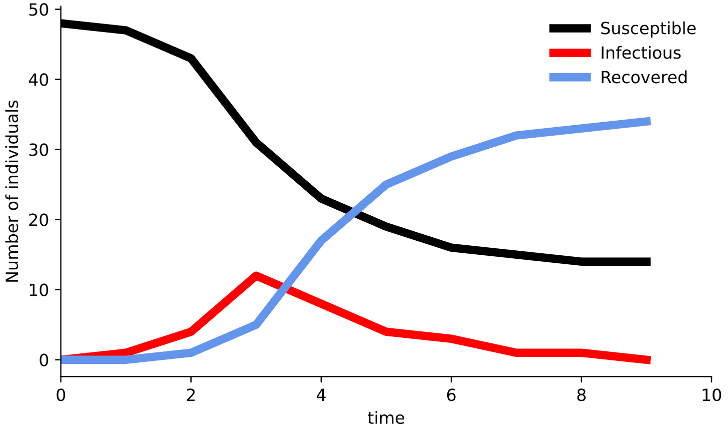
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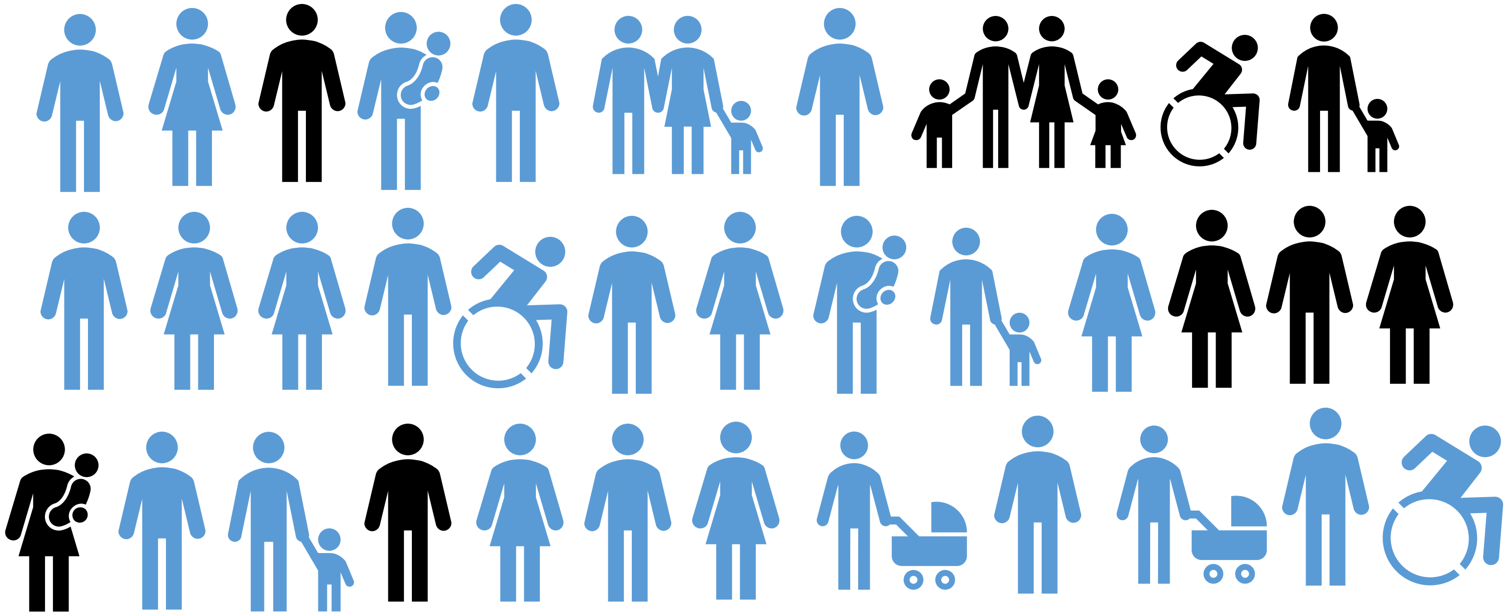
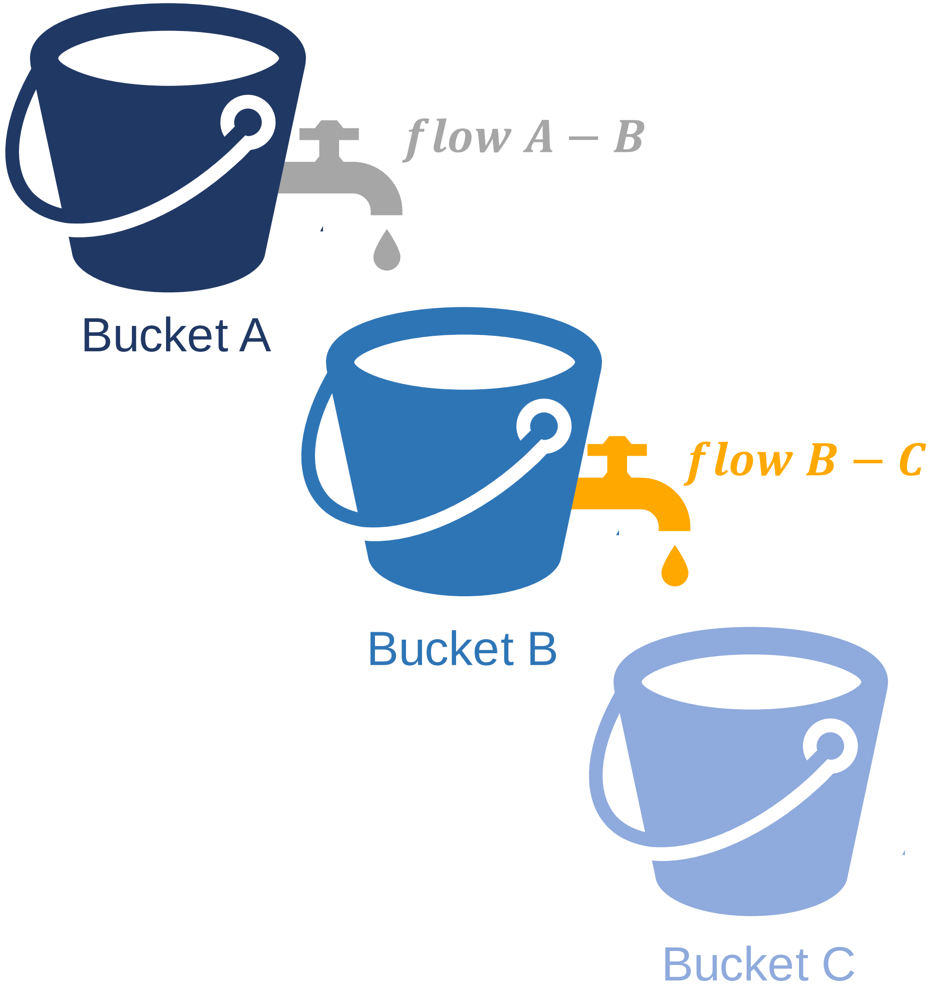


Compartmental models



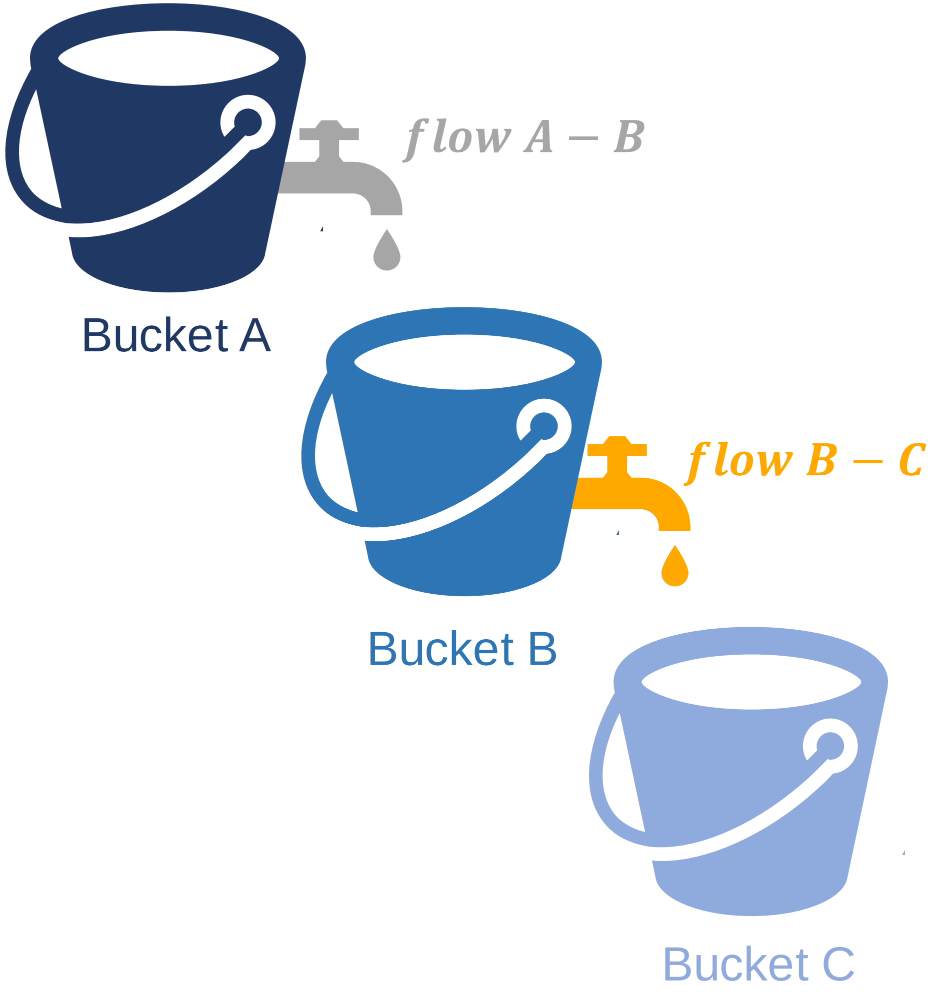
Difference equations

Bucket A (t+1) = Bucket A (t) – flow A-B

Bucket B (t+1) = Bucket B (t) + flow A-B – flow B-C

Bucket C (t+1) = Bucket C (t) + flow B-C

Compartmental models



Difference equations

Bucket A (t+1) = Bucket A (t) – flow A-B

Bucket B (t+1) = Bucket B (t) + flow A-B – flow B-C

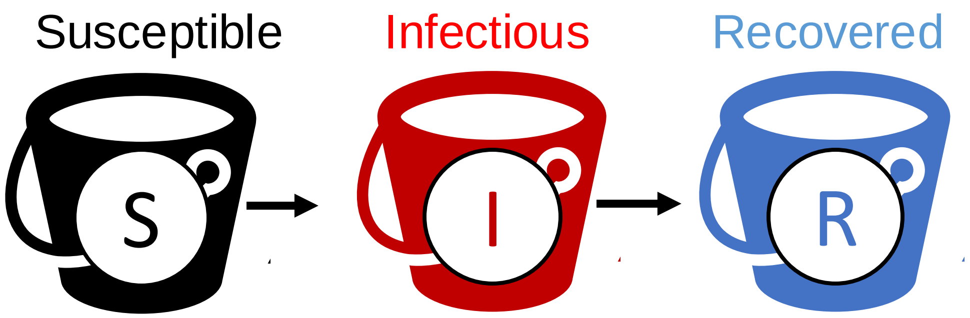
Bucket C (t+1) = Bucket C (t) + flow B-C

Differential equations

$$\frac{dA}{dt} = -a\,A$$

$$\frac{dB}{dt} = a\,A - b\,B$$

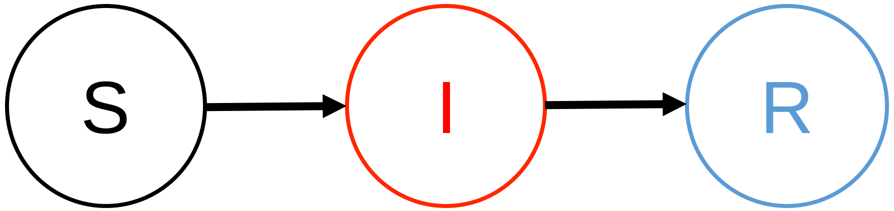
$$\frac{dC}{dt} = b\,B$$



$$\frac{dS}{dt} = -(\text{rate out})S$$

$$\frac{dI}{dt} = (\text{rate in})S - (\text{rate out})I$$

$$\frac{dR}{dt} = (\text{rate in})I$$



$$-(\text{rate out})S$$

$$\frac{dS}{dt} = -(\text{infectiousness} * P(\text{contact with infectious person}))S$$

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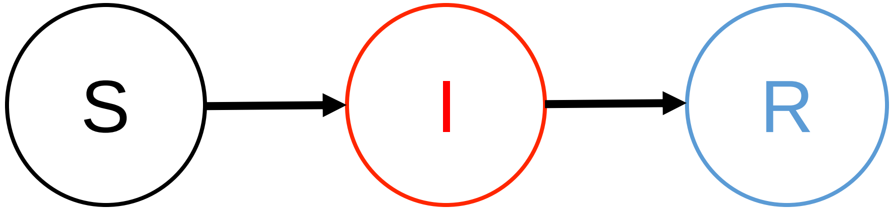


$$-(\text{rate out})S$$
$$-(\text{infectiousness} * P(\text{contact with infectious person}))S$$

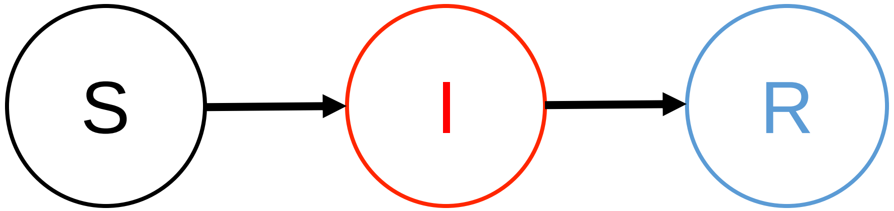
$$\frac{dS}{dt} = -\beta \frac{I}{N} S$$

β : *infectiousness*

N : *population size* → $N = S+I+R$

$$\frac{dI}{dt} = (\text{rate in})S - (\text{rate out})I$$

$$\frac{dR}{dt} = (\text{rate in})I$$



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$$-(\text{infectiousness} * P(\text{contact with infectious person}))S$$

$$\frac{dS}{dt} = -\beta \frac{I}{N} S$$

$$(\text{rate in})S - (\text{rate out})I$$

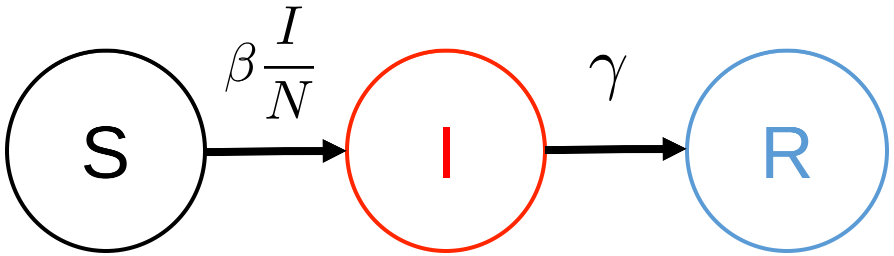
$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I$$

$$\frac{dR}{dt} = (\text{rate in})I$$

β : *infectiousness*

N : *population size* → $N = S+I+R$

γ : *recovery*



$-(\text{rate out})S$

$-(\text{infectiousness} * P(\text{contact with infectious person}))S$

$$\frac{dS}{dt} = -\beta \frac{I}{N} S$$

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$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I$$

$(\text{rate in})I$

$$\frac{dR}{dt} = \gamma I$$

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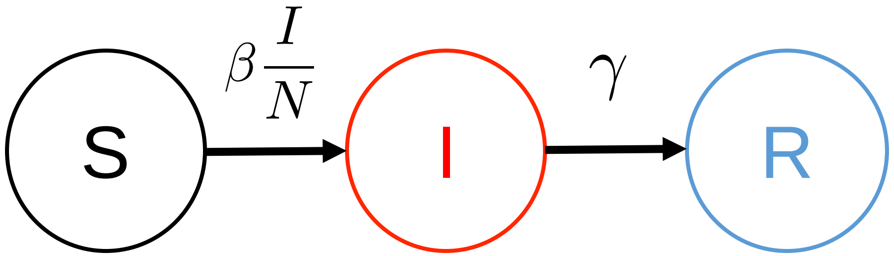

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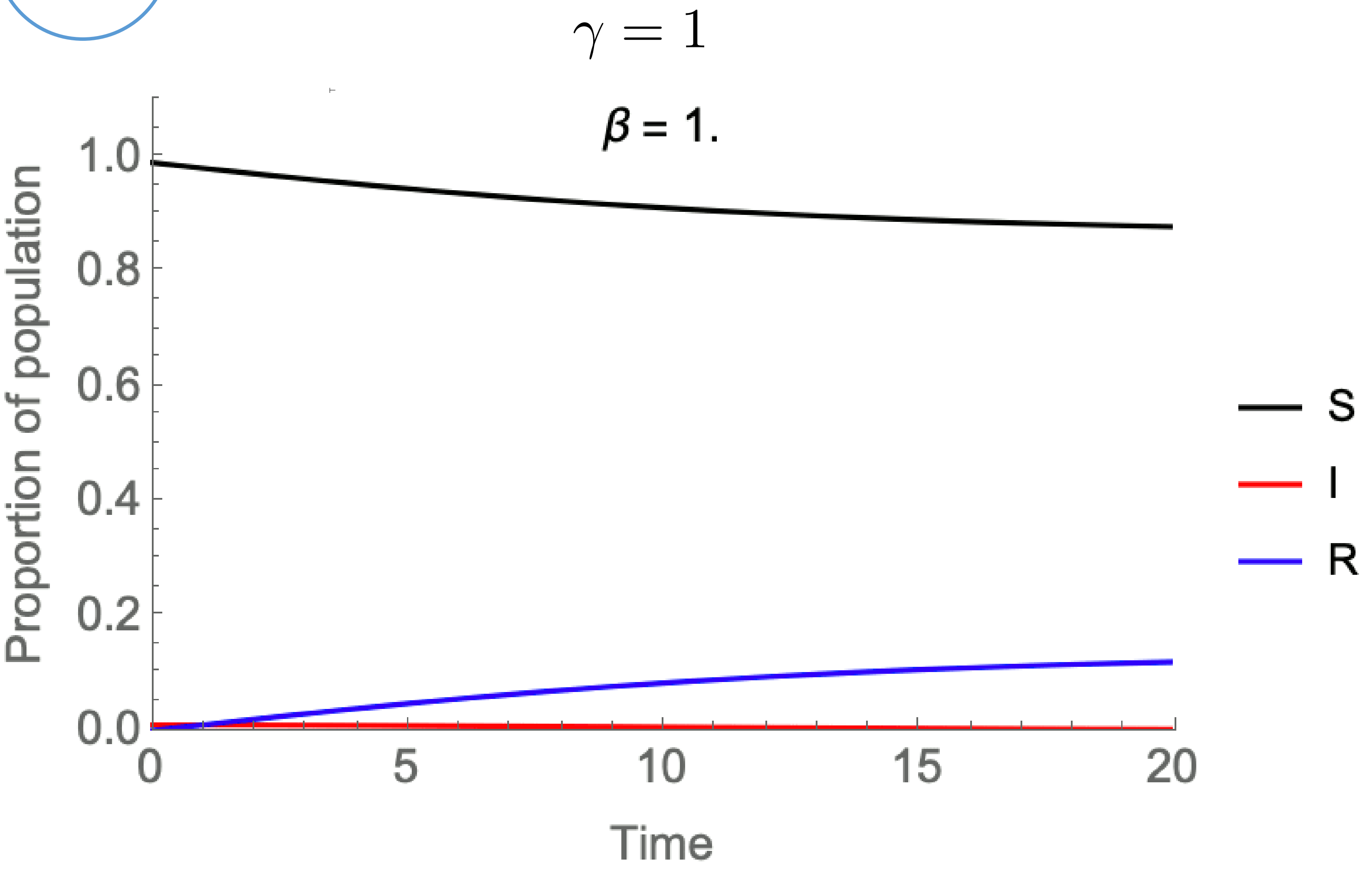
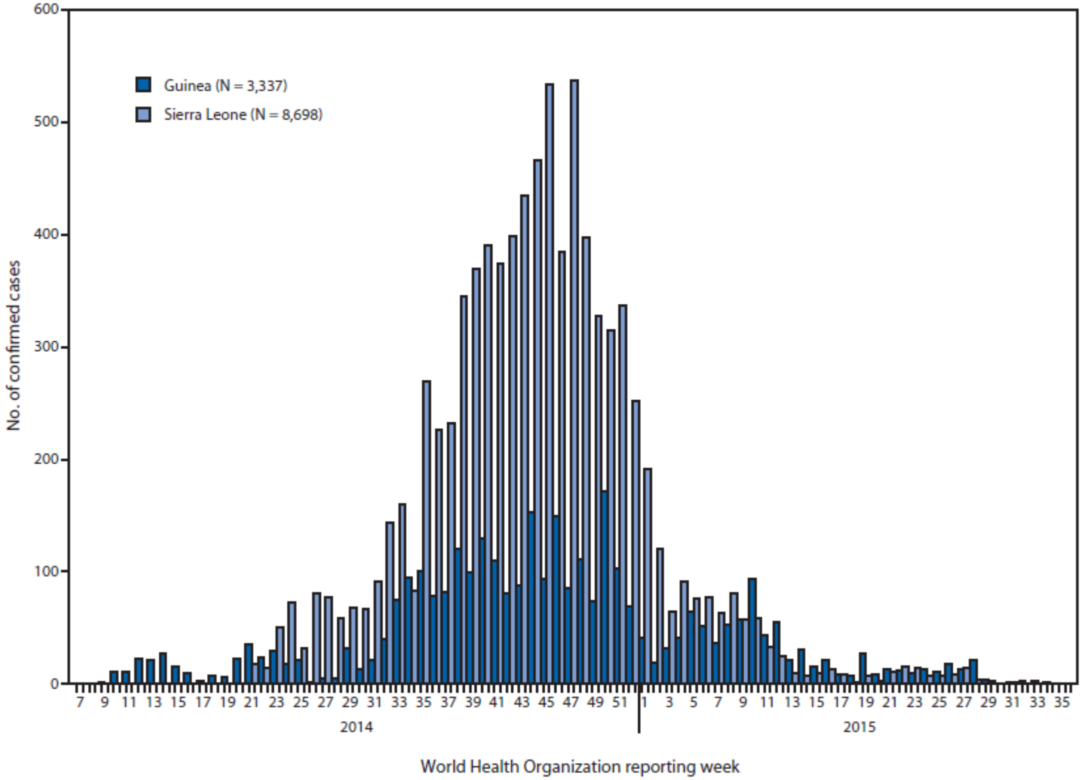
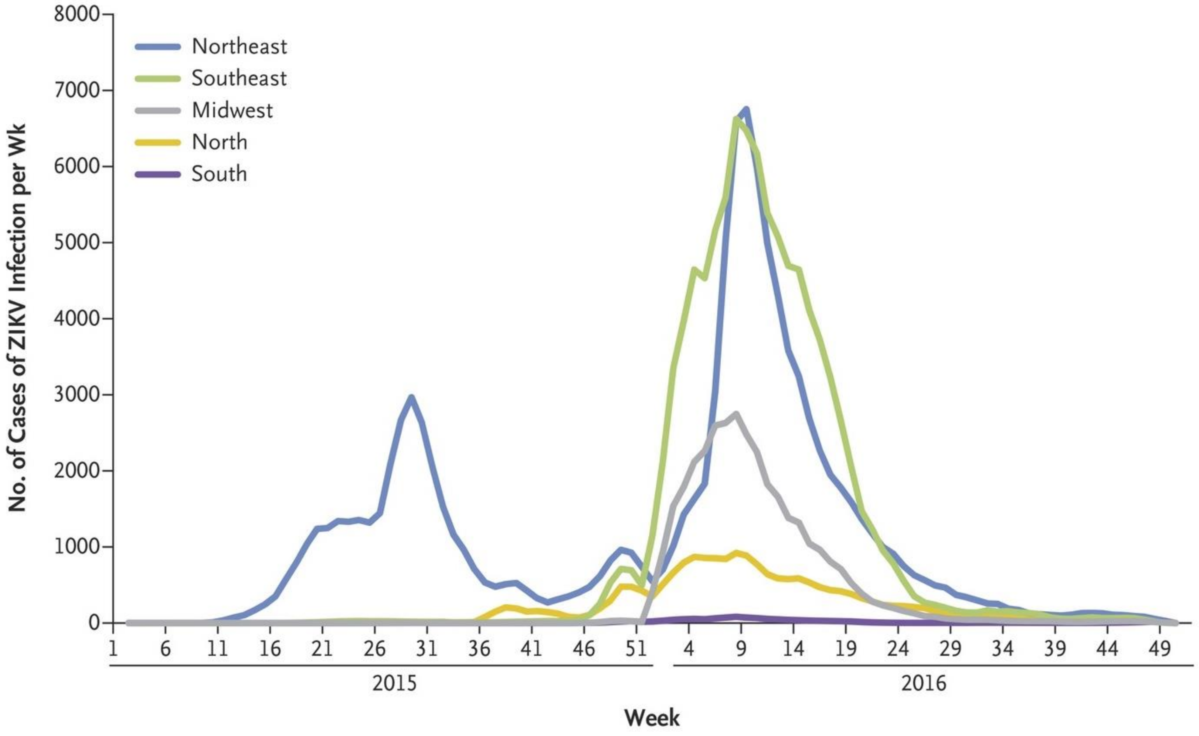


Ebola: weekly confirmed cases in Guinea and Sierra Leone in 2014-15



Zika: weekly suspected cases in different regions of Brazil in 2015-16

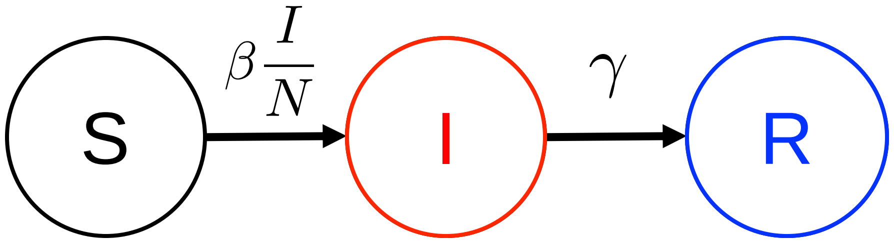


Sources: CDC, NEJM: 10.1056/NEJMc1608612

Will there be an outbreak?

Will an outbreak happen?

“The number infected must increase”



$$\frac{dS}{dt} = -\beta \frac{I}{N} S$$

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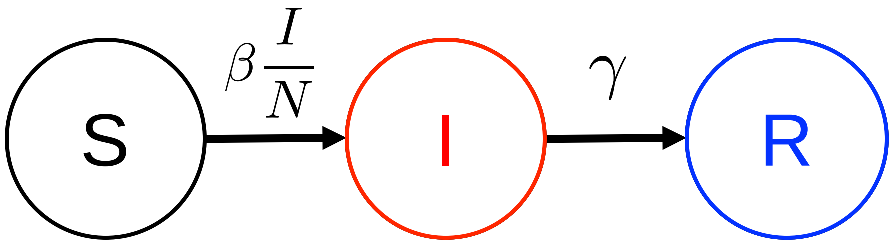
$$\frac{dR}{dt} = \gamma I$$

Will an outbreak happen?



$$\frac{dS}{dt} = -\beta \frac{I}{N} S$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

"The number infected must increase"

$$\frac{dI}{dt} > 0$$

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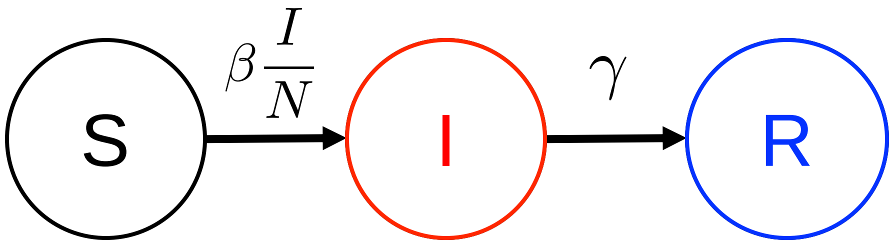

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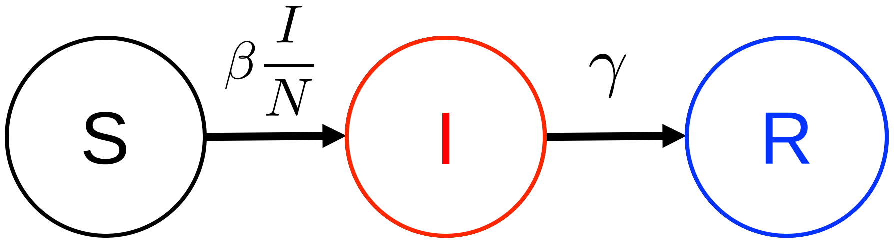

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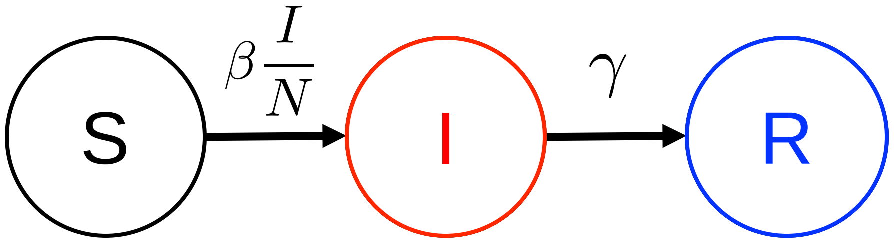

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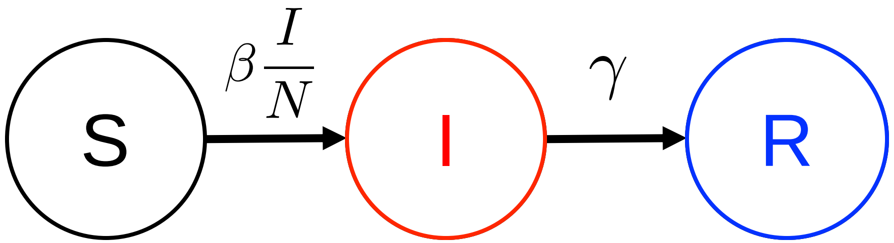
Will an outbreak happen?



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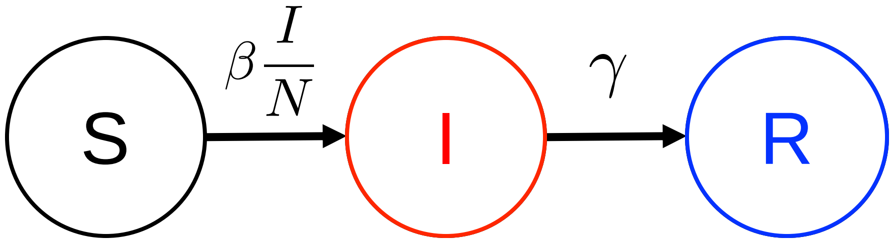
infectiousness $\beta > \gamma$ *recovery*

Will an outbreak happen?



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infectiousness $\beta > \gamma$ *recovery*

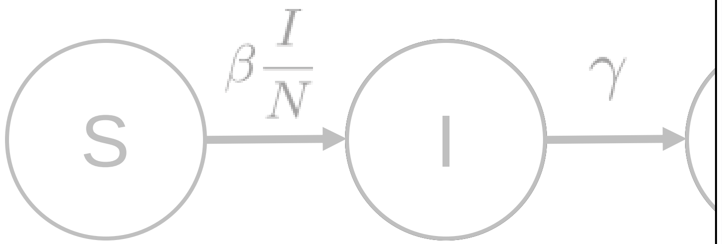
Basic reproduction number

$$R_0 = \boxed{\frac{\beta}{\gamma} > 1}$$

$\frac{\beta}{\gamma}$: average number of infections caused by one infectious person

Will an outbreak happen?

infected must increase”



$$\frac{dS}{dt} = -\beta \frac{I}{N} S$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

$I > 0$

$I \qquad (S_0 \approx N)$

Values of R_0 of well-known infectious diseases[1]

Disease	Transmission	R_0
Measles	Airborne	12–18
Diphtheria	Saliva	6-7
Smallpox	Airborne droplet	5–7
Polio	Fecal-oral route	5–7
Rubella	Airborne droplet	5–7
Mumps	Airborne droplet	4–7
HIV/AIDS	Sexual contact	2–5
Pertussis	Airborne droplet	5.5[2]
SARS	Airborne droplet	2–5[3]
Influenza (1918 pandemic strain)	Airborne droplet	2–3[4]
Ebola (2014 Ebola outbreak)	Bodily fluids	1.5-2.5[5]

Source: Wikipedia

β: infectiousness

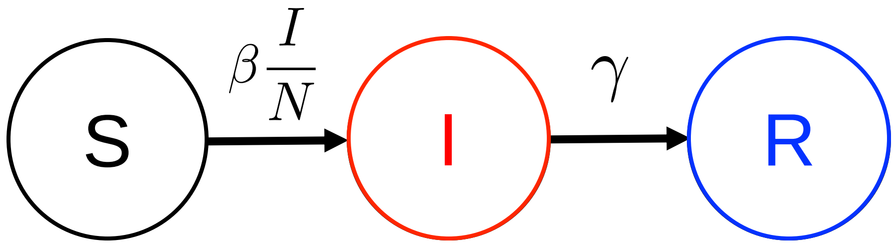
γ: rate of recovery

$\frac{\beta}{\gamma}$: average number of infections caused by one infectious person

Can we prevent the outbreak?

Can we prevent it?

What proportion, p, should we vaccinate?



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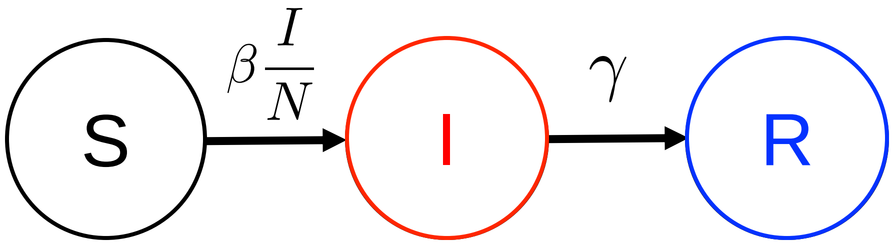

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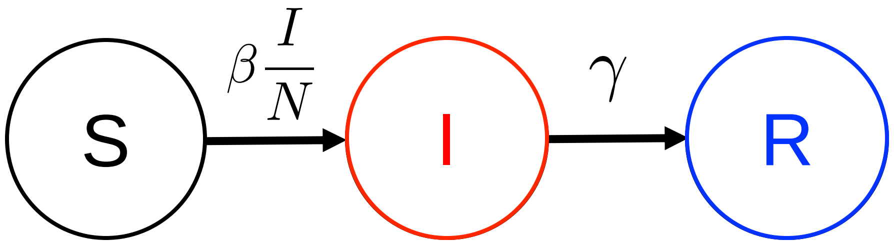

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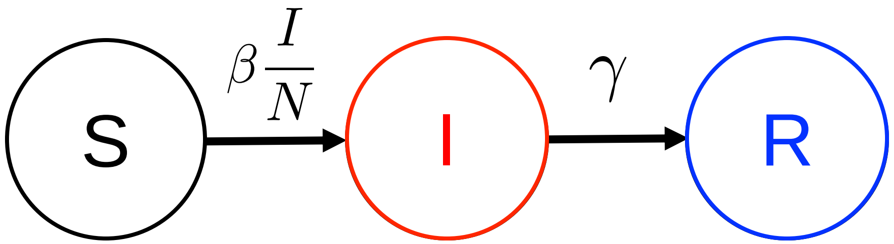

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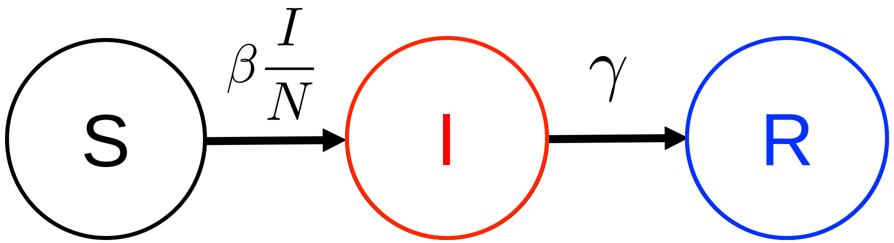

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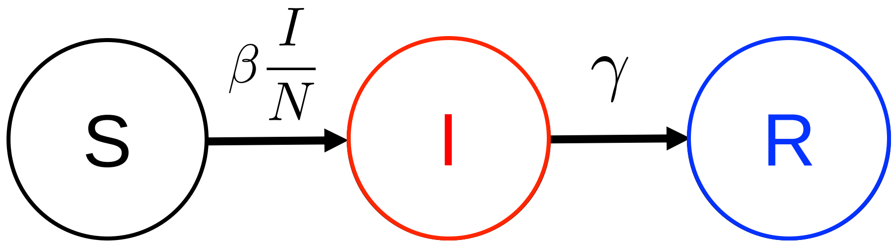
$$\beta\frac{I}{N}S < \gamma I$$

$$\beta\frac{I}{N}(1-p)N < \gamma I \qquad (S_0 \approx (1-p)N)$$

Can we prevent it?



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$$\beta(1-p) < \gamma$$

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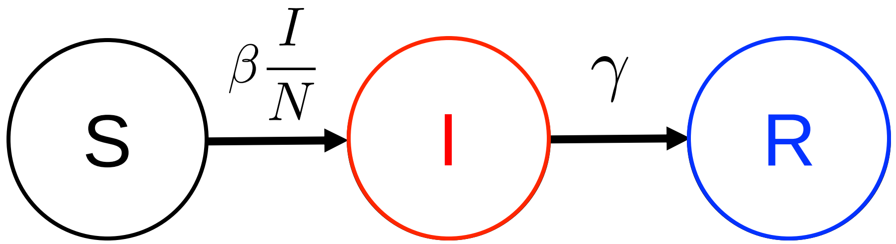

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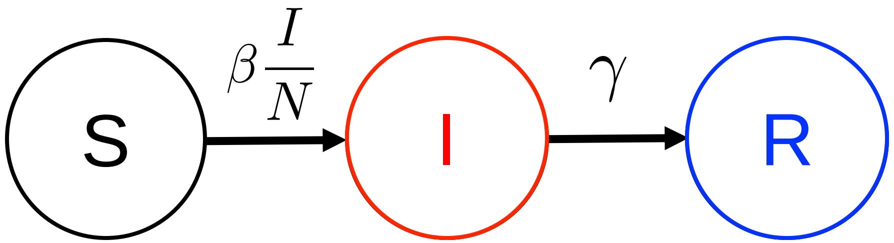

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$$R_0 < \frac{1}{(1-p)}$$

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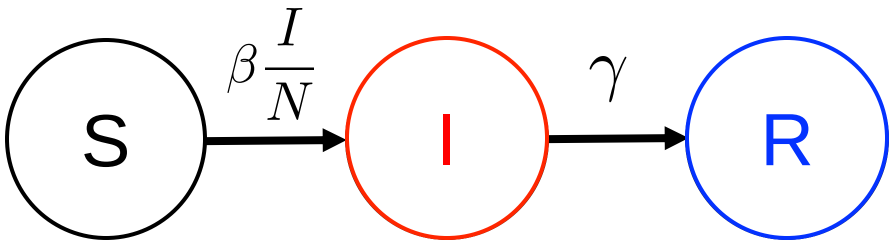

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$$\beta(1-p) < \gamma$$

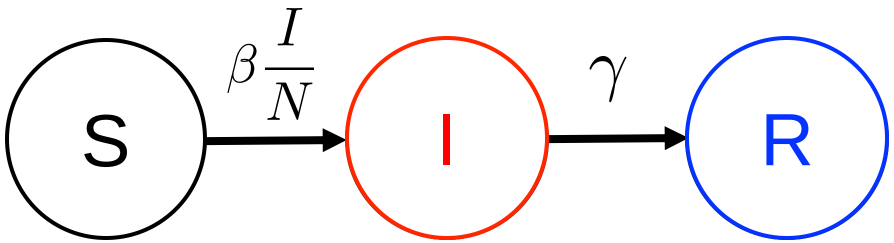
$$\frac{\beta}{\gamma} < \frac{1}{(1-p)}$$

$$R_0 < \frac{1}{(1-p)}$$

$$p > 1 - \frac{1}{R_0}$$

Critical vaccination threshold

Can we prevent it?



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$$\beta (1-p) < \gamma$$

$$\frac{\beta}{\gamma} < \frac{1}{(1-p)}$$

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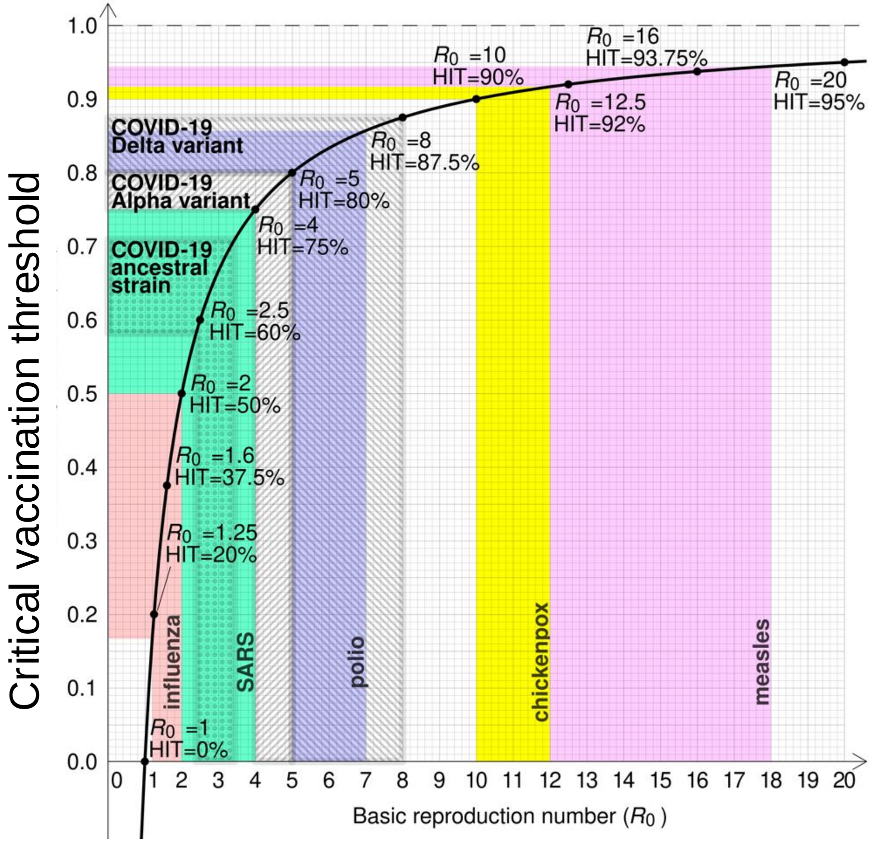


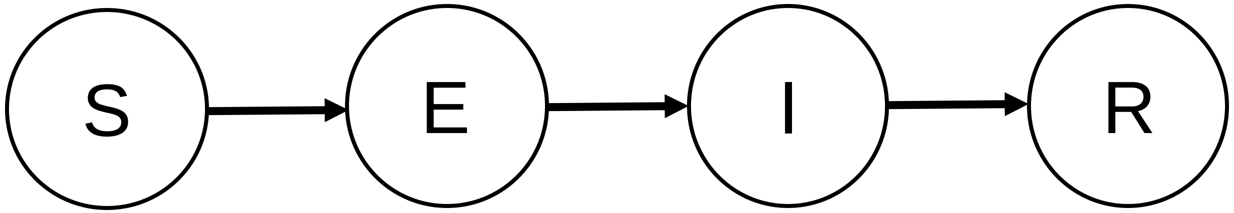
Extensions of the SIR model

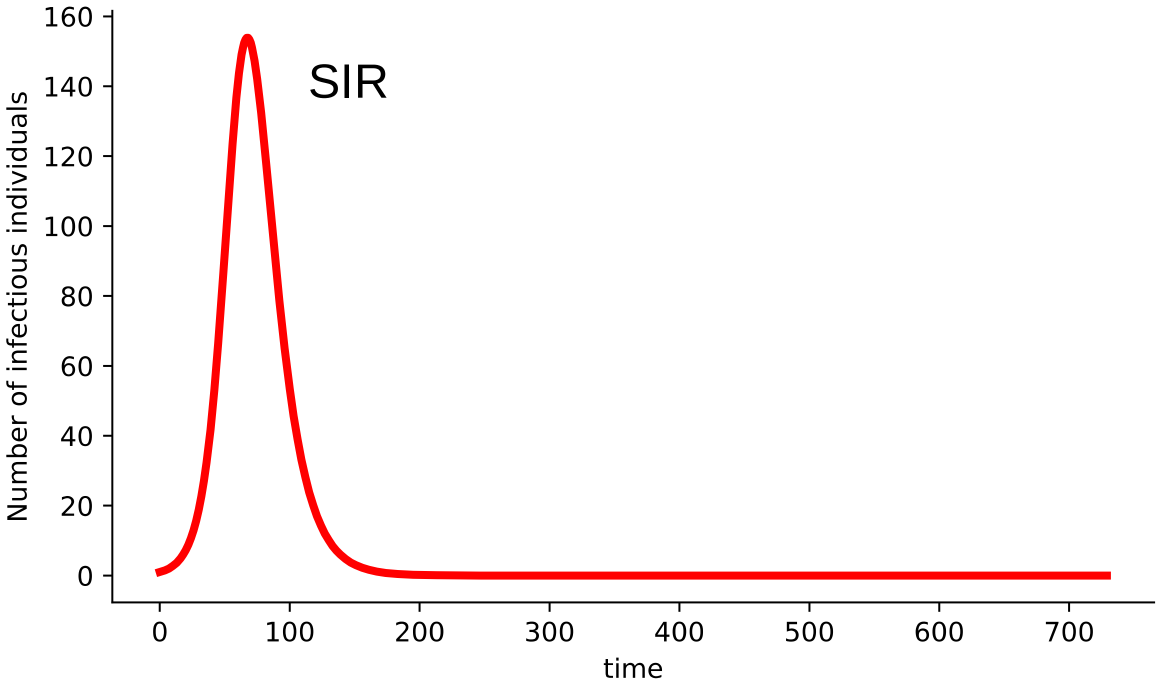
We can increase model complexity and realism by:

- adding **disease states** (compartments)
- changing **transitions** (flows), or
- splitting compartments to account for **population heterogeneity**

Making compartmental models (a bit) more realistic

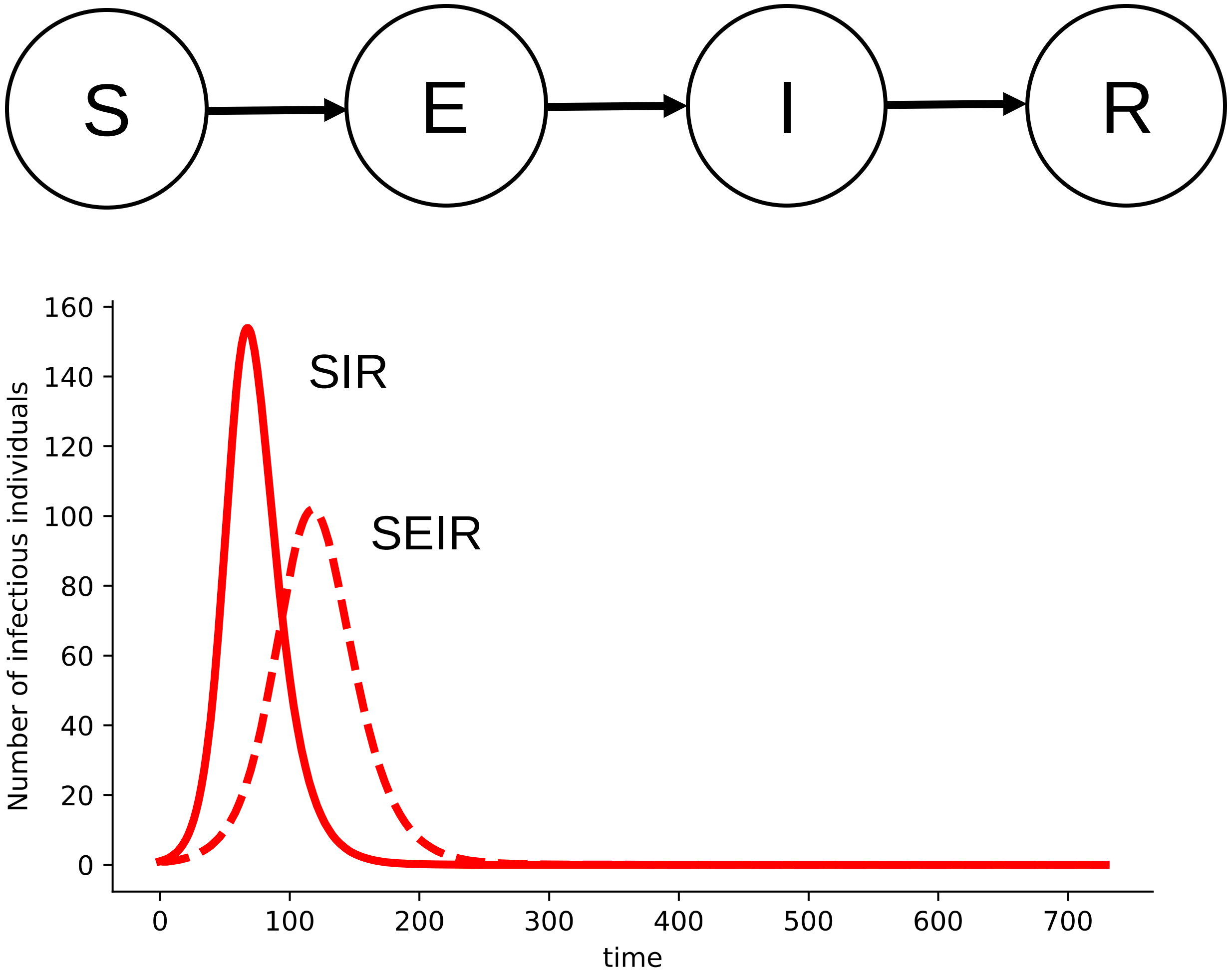
What happens if there's an incubation period?





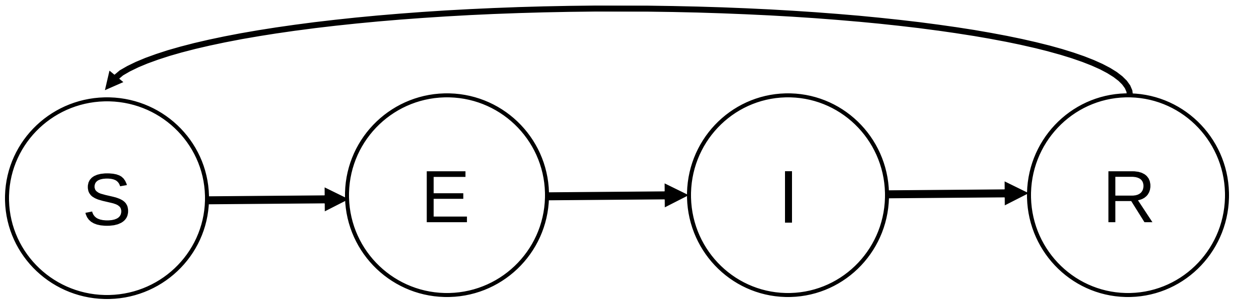
Making compartmental models (a bit) more realistic

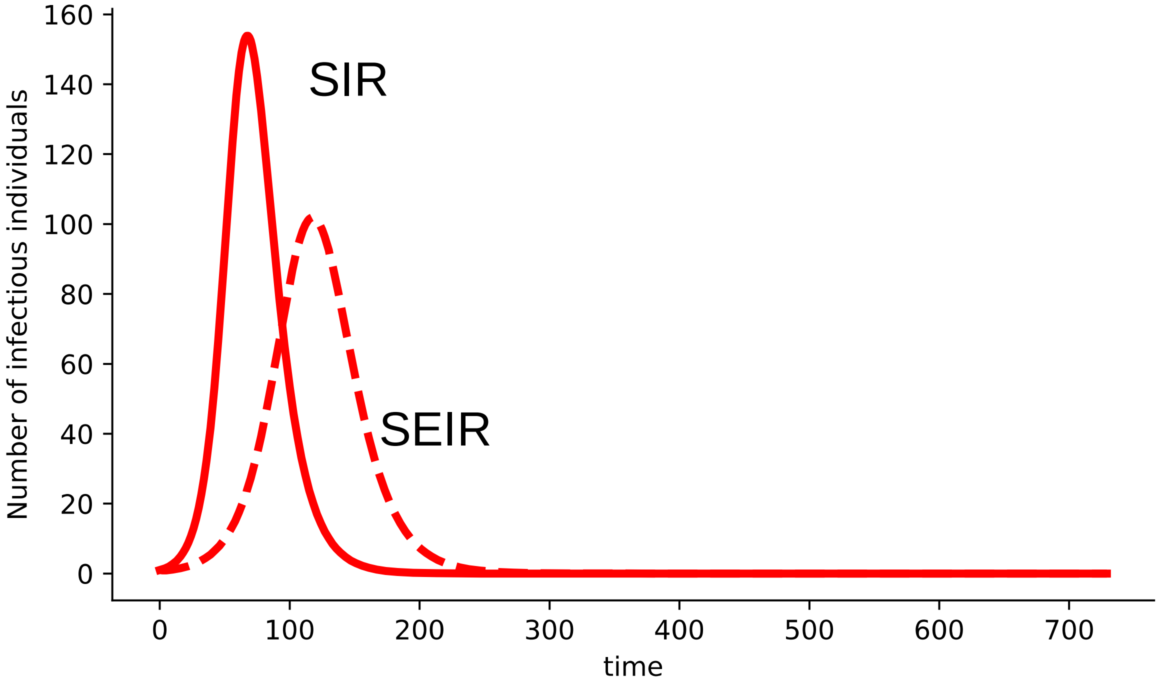
What happens if there's an incubation period?



Making compartmental models (a bit) more realistic

What happens if you lose immunity?





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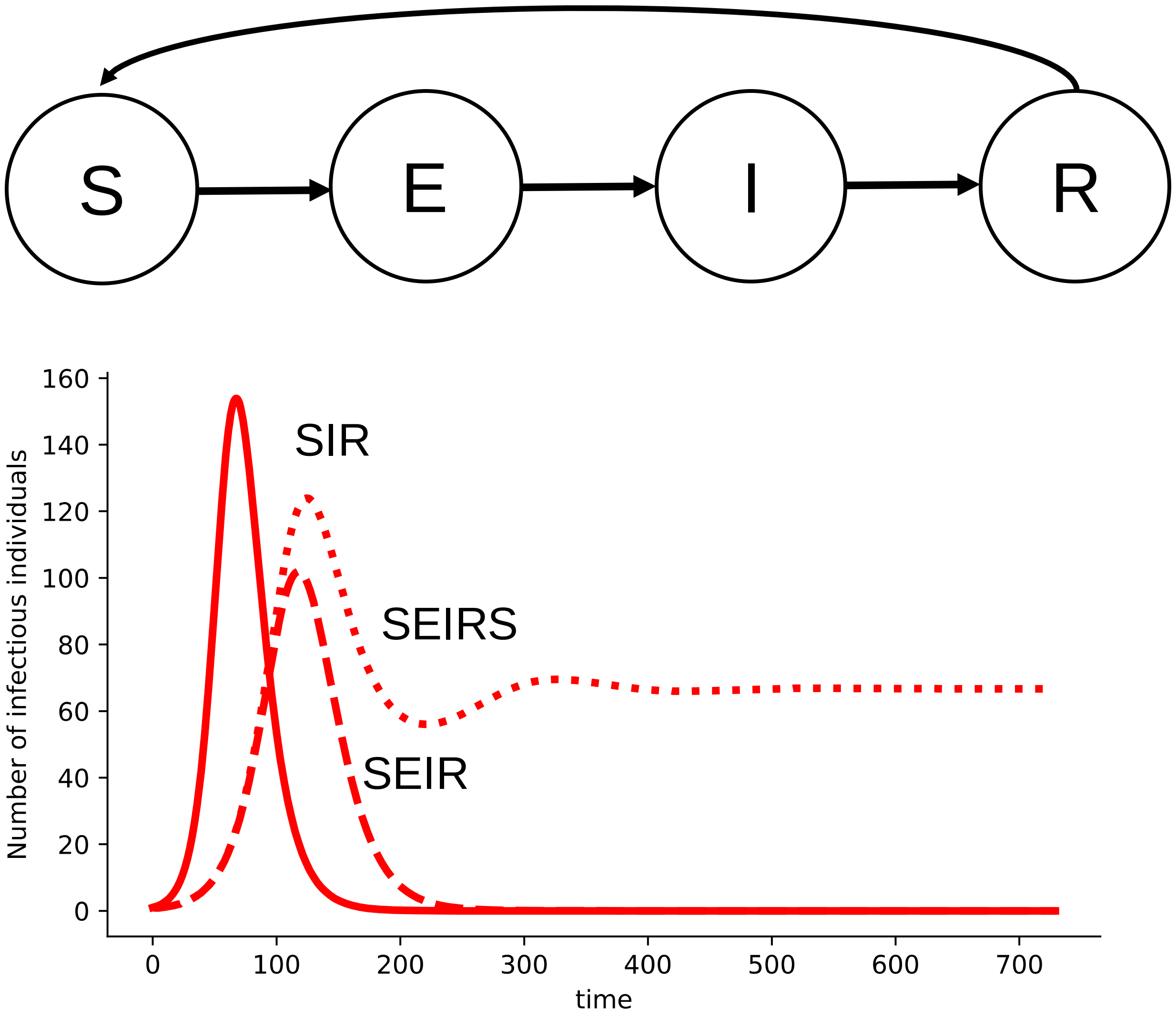
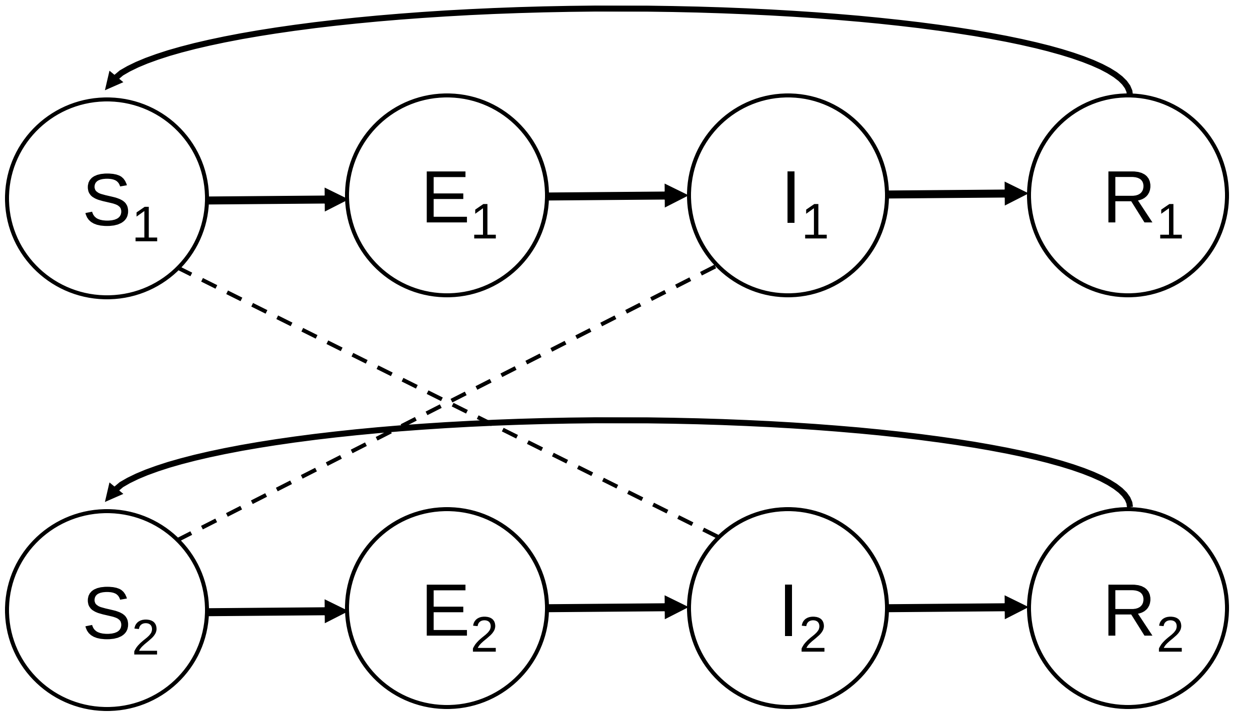


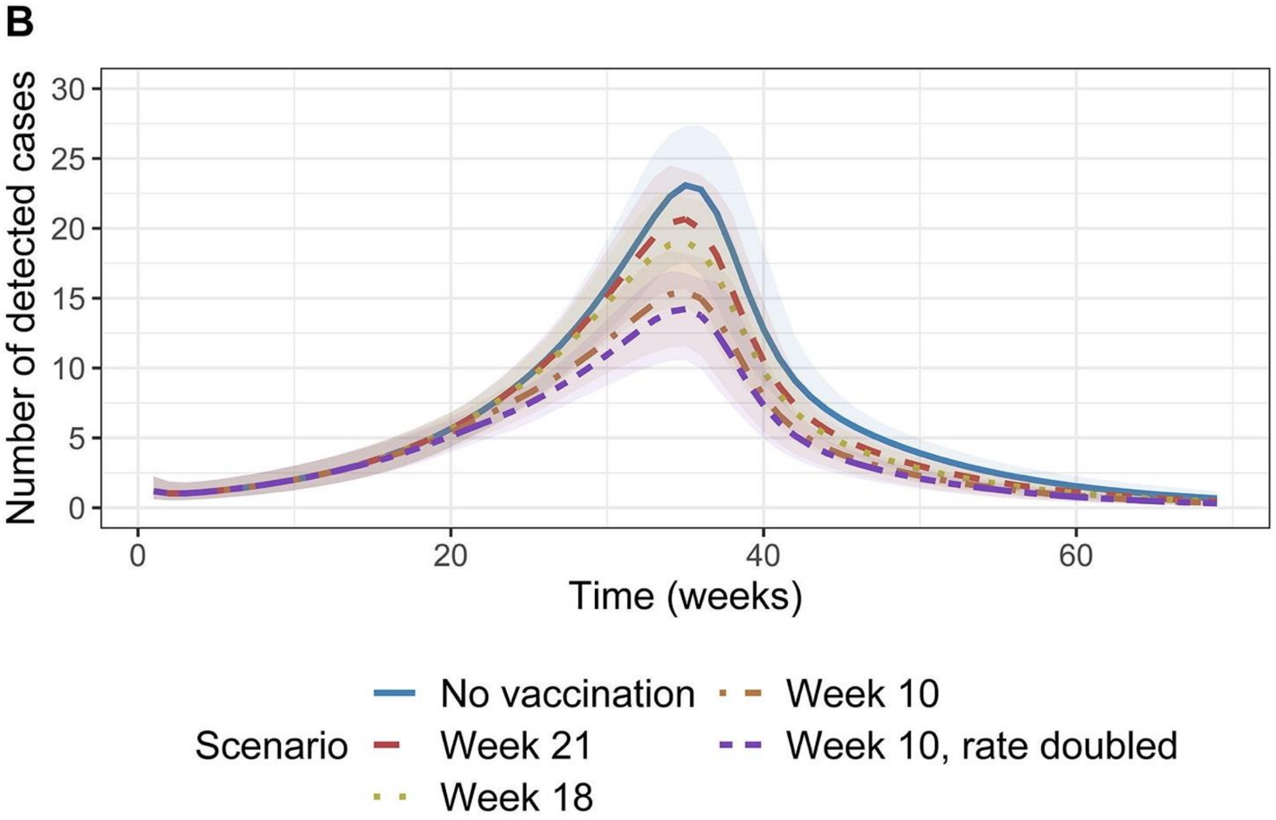
Making compartmental models (a bit) more realistic

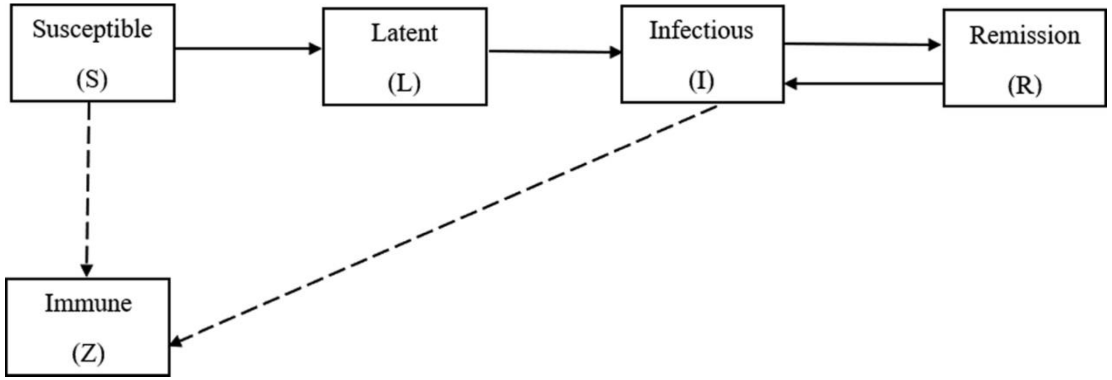
What if age structure matters?



We do really use this stuff...

Comparing timing and coverage of Hepatitis A vaccine

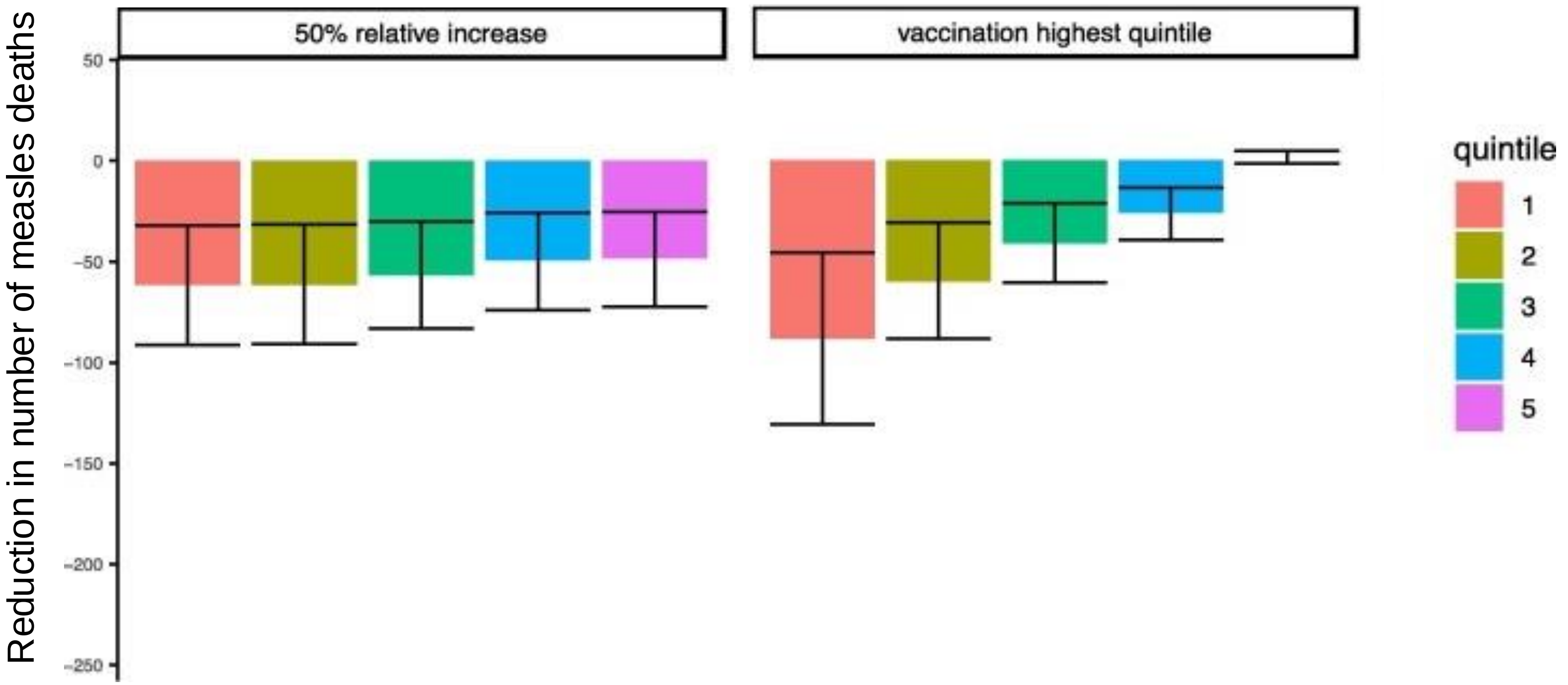




https://pubmed.ncbi.nlm.nih.gov/34686394/

Dankwa, *et al.* (2021)

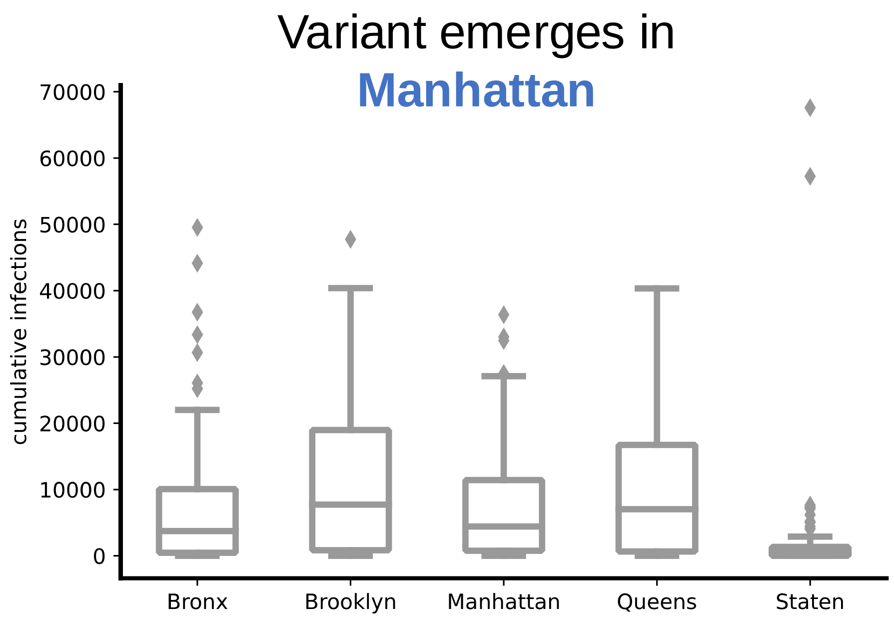
Incorporating equity in infectious disease modeling and vaccination decisions

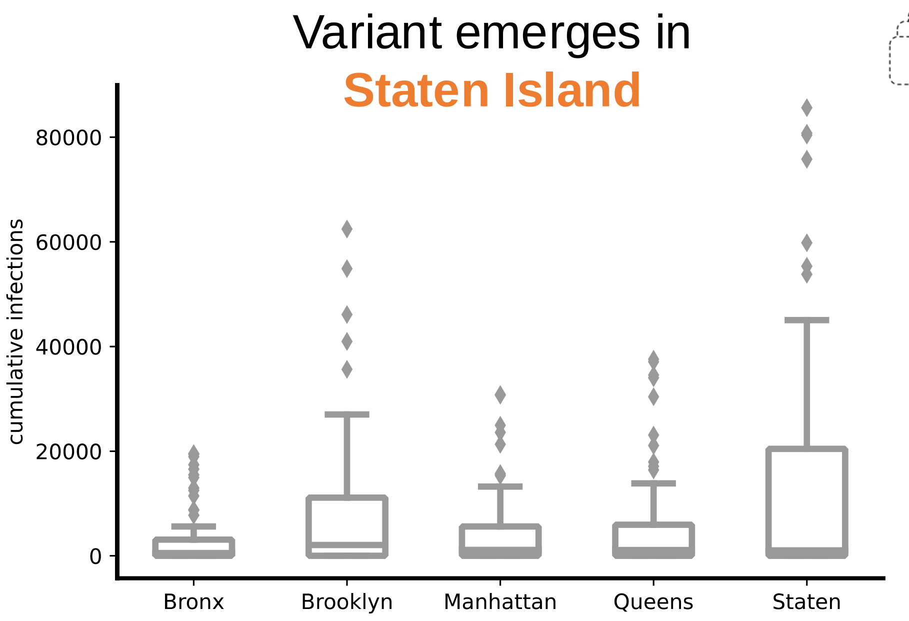


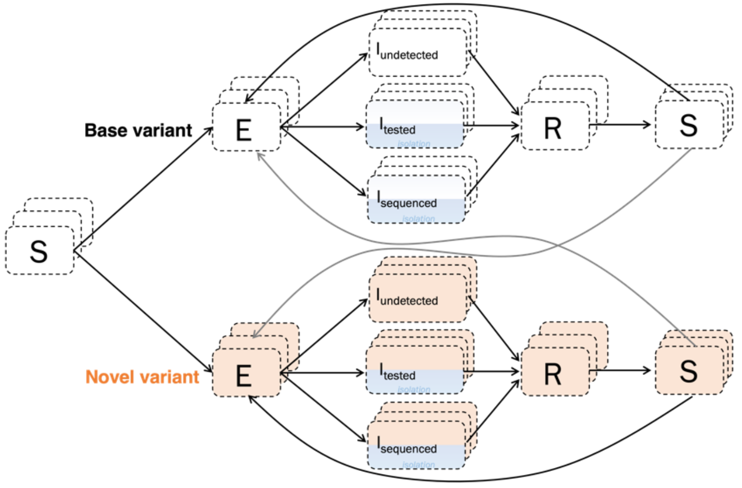
https://pubmed.ncbi.nlm.nih.gov/33863575/

Menkir *et al.* (2021)

Where do we see the greatest number of infections if new SARS-CoV-2 variants emerge in different places across the city?







https://journals.plos.org/ploscompbiol/article?id=10.1371/journal.pcbi.1012416

Oliveira Roster et al. (2024)

Final thoughts

- Compartmental models are ***simple*** but ***powerful***
- Start by understanding the disease process
- Identify the public health goal
- Translate the disease process into a model
- Start with a simple model, add complexity as needed, but no more!
- Return to the disease process & public health impact

Thank you!

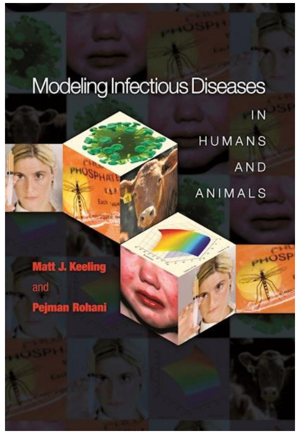
Textbooks & Academic Articles

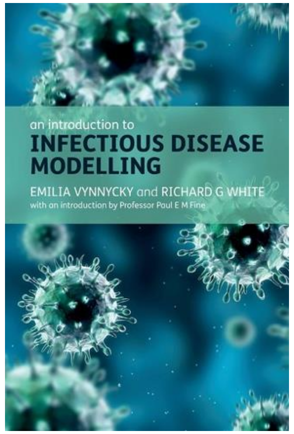
Modeling Infectious Diseases in Humans and Animals
(Matt J. Keeling and Pejman Rohani)

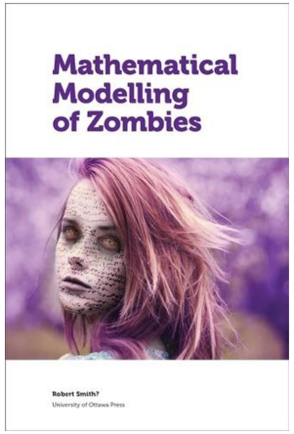
An Introduction to Infectious Disease Modelling
(Emilia Vynnycky and Richard White)

Mathematical Modelling of Zombies
(Robert Smith?)
https://people.maths.ox.ac.uk/maini/PKM%20publications/384.pdf

An introduction to compartmental modeling for the budding infectious disease modeler
(Lauren Childs)
https://vtechworks.lib.vt.edu/items/61e9ca00-ef21-4356-bcd7-a9294a1d2f17



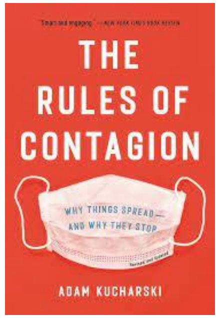


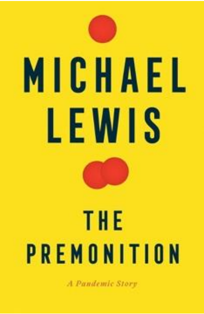


Popular Science

The Rules of Contagion: Why Things Spread and Why They Stop
(Adam Kucharski)

The Premonition: A Pandemic Story
(Michael Lewis)





Online Courses

Introduction to Infectious Disease Modelling
(Caroline Buckee, Inga Holmdahl, Ayesha Mahmud)
https://ccdd.hsph.harvard.edu/introduction-to-infectious-disease-modeling/

Coursera: Infectious Disease Modelling Specialization
(Nimalan Arinaminpathy)
https://www.coursera.org/specializations/infectious-disease-modelling#courses

Contagious Maths
(Julia Gog)
https://plus.maths.org/content/contagious-maths